COMPLEX INTEGRATION :

Basics:-

A pair of equations of the form $x = x(t)$, $y = y(t)$. Where t is a real parameter represents a curve 'c' in the xy-plane.

The equation $z = z(t) = x(t) + iy(t)$ represents the equation of the curve c in the complex form where t varies over an appropriate interval.

Ex: $z = ae^{it}$; $0 \leq t \leq 2\pi \longrightarrow$ circle centred at origin & radius a.

SIMPLE CURVE:-

A curve C is said to be simple if it does not intersect itself.

ie; The curve C given by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$ is simple if $z(t_1) \neq z(t_2)$ for any two different values t_1 and t_2 of t in (a,b).

Ex: Semi circle lies above the real axis.

SIMPLE CLOSED CURVE :-

A curve C is said to be a simple closed curve if it is simple and the end points coincide.

; The equation $z(t) = x(t) + iy(t)$, $a \leq t \leq b$ and $z(t_1) \neq z(t_2$

| $z(a) = z(b)$

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angle etc.

SMOOTH CURVE :-

A curve C is said to be smooth if there exists a unique tangent at each of its points (differentiable curves).

Ex: Arc of a circle is a smooth curve

Triangle is not a smooth curve.

CONTOUR :-

A continuous chain of a finite number of smooth curves is called a contour.

Ex: Circle is contour (single smooth curve)

Triangle is contour (chain of three lines)

Rectangle is contour (Chains of four lines).

POSITIVELY ORIENTED CURVE :

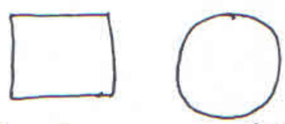
A curve with positive orientation is called a positively oriented curve and a curve with negative orientation is called a negatively oriented curve.

Positive orientation → If the point $z(t)$ on the curve $z(t) = x(t) + iy(t)$, $a \leq t \leq b$ varies from the initial point to the Terminal point as t increases from a to b.

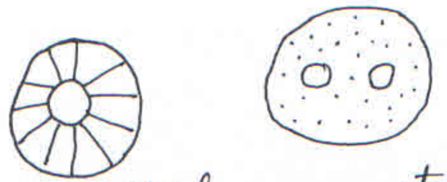
SIMPLY CONNECTED REGION :-

A region D in the complex plane is said to be simply connected if every simple closed curve lying entirely in D can be shrunk to a point without crossing the boundary of D.

A region which is not simply connected is called a multiple connected region (region having holes).



Simply connected



Multiple connected

LINE INTEGRAL OF A COMPLEX FUNCTION :-

The line integral of a complex function $f(z)$ from z_1 to z_2 along a curve c is defined by

$$\int_c f(z)\,dz = \int_c (u+iv)\,(dx+idy) = \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy)$$

where c is called the path of integration. If c is a closed curve then the line integral is denoted by $\oint_c$

PROPERTIES :-

(i) $\int_c (K_1 f(z) + K_2 g(z))\,dz = K_1 \int_c f(z)\,dz + K_2 \int_c g(z)\,dz$

(ii) $\int_{-c} f(z)\,dz = -\int_c f(z)\,dz$

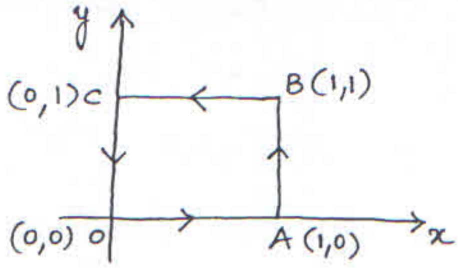
(iii) $\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz$

(iv) $\left|\int_c f(z)\,dz\right| \leq \int_c |f(z)|\,|dz|$.

PROBLEMS :-

1. Evaluate $\oint_C |z|^2 dz$ around the square region with vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

Sol: The region C is given by



$$\int_C |z|^2 dz = \int_{OA} |z|^2 dz + \int_{AB} |z|^2 dz + \int_{BC} |z|^2 dz + \int_{CO} |z|^2 dz \longrightarrow (1)$$

Along OA : $y=0$ and $z = x+iy = x$, x varies from 0 to 1.

$$\therefore \int_{OA} |z|^2 dz = \int_{x=0}^{1} (x^2+y^2)(dx+idy) = \int_{x=0}^{1} x^2 dx = \frac{x^3}{3} = \frac{1}{3}$$

Along AB : $x=1$, $dx=0$, $z = x+iy = 1+iy$, y varies from 0 to 1

$$\therefore \int_{AB} |z|^2 dz = \int_{y=0}^{1} (x^2+y^2)(dx+idy) = \int_{y=0}^{1} (1+y^2)(idy) = i\int_{y=0}^{1} (1+y^2) dy = \frac{4}{3} i .$$

Along BC : $y=1$, $dy=0$, $z = x+iy = x+i$, x varies from 1 to 0.

$$\therefore \int_{BC} |z|^2 dz = \int_{x=1}^{0} (x^2+y^2)(dx+idy) = -\int_{x=0}^{1} (1+x^2) dx = -\left(\frac{1}{3}+1\right) = \frac{-4}{3}$$

Along CO : $x=0$, $z = x+iy = iy$, y varies from 1 to 0.

$$\therefore \int_{CO} |z|^2 dz = \int_{1=1}^{0} (x^2+y^2)(dx+idy) = -\int_{y=0}^{1} y^2 (idy) = -i\int_{y=0}^{1} y^2 dy = \frac{-i}{3} .$$

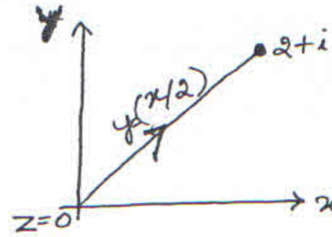
$$\therefore \oint_C |z|^2 dz = \frac{1}{3} + \frac{4}{3} i - \frac{4}{3} - \frac{i}{3} = -1+i .$$

[2] Evaluate $\int_{z=0}^{2+i} (\bar{z})^2 dz$ along the following curves.

(i) The straight line $y = x/2$

(ii) The real axis from 0 to z and then vertically to $2+i$.

<u>Sol</u>: (i) <u>Along $y = x/2$</u>:



We have $x = 2y \Rightarrow dx = 2dy$

z varies from 0 to $2+i \Rightarrow y$ varies from 0 to 1.

$$\therefore \int_{z=0}^{2+i} (\bar{z})^2 dz = \int_{y=0}^{1} (x-iy)^2 (dx+idy)$$

$$= \int_{y=0}^{1} (x^2-y^2-2ixy)(2dy+idy)$$

$$= \int_{y=0}^{1} (3y^2-4y^2 i)(2+i)\,dy$$

$$= (2+i)\int_{y=0}^{1} (3y^2 - i4y^2)\,dy$$

$$= \frac{5}{3}(2-i)$$

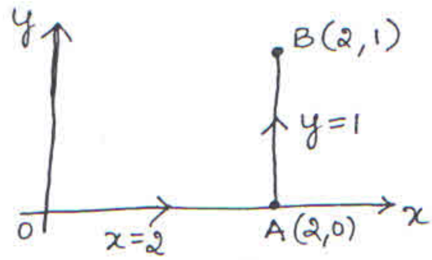
(OR)

$$\int_{y=0}^{1} (2y-iy)^2 (2dy+idy)$$

$$= \int_{y=0}^{1} (2-i)^2 y^2 (2+i)\,dy$$

$$= (2-i)^2 (2+i) \int_{y=0}^{1} y^2 dy$$

(ii) Here the path of integration is given by



$$\therefore \int_{z=0}^{2+i} (\bar{z})^2 dz = \int_{OA} (\bar{z})^2 dz + \int_{AB} (\bar{z})^2 dz \longrightarrow (1)$$

<u>Along OA</u>: $y=0$, $z = x+iy = x$, $dz = dx$, x varies from 0 to 2

$$\therefore \int_{OA} (\bar{z})^2 dz = \int_{x=0}^{2} (x-iy)^2 (dx+idy) = \int_{x=0}^{2} x^2 dx = \frac{8}{3}$$

Along AB : $x=2$, $z=x+iy=2+iy$, $dz=i\,dy$, y varies from 0 to 1.

$$\therefore \int_{AB} (\bar{z})^2 dz = \int_{y=0}^{1} (x-iy)^2 (dx+i\,dy) = \int_{y=0}^{1} (2-iy)^2 (i\,dy)$$

$$= i\int_{y=0}^{1} (4-y^2-4iy)\,dy = i(4-2i-\tfrac{1}{3}) = 4i+2-\tfrac{1}{3}i.$$

$\therefore$ From (1),

$$\int_{z=0}^{2+i} (\bar{z})^2 dz = \frac{8}{3}+2+i(4-1/3) = \frac{1}{3}(14+11i).$$

$\boxed{3}$ Evaluate $\int_C (x^2+ixy)\,dz$ from $A(1,1)$ to $B(2,8)$ along $x=t$ and $y=t^3$.

$$\therefore \int_C (x^2+ixy)\,dz = \int_C (x^2+ixy)(dx+i\,dy)$$

$$= \int_C (x^2dx - xy\,dy) + i\int_C (xy\,dx + x^2dy)$$

$$= \int_{t=1}^{2} t^2dt - 3t^6dt + i\int_{t=1}^{2} 4t^4dt$$

$$= -\frac{1094}{21} + \frac{124i}{5}$$

$\boxed{4}$ Evaluate $\int_0^{1+i} z^2 dz$ along $y=x^2$

Sol : Along $y=x^2$, $dy=2x\,dx$

x varies from 0 to 1.

$$\therefore \int_{z=0}^{1+i} z^2 dz = \int_{z=0}^{1+i} (x^2-y^2+2ixy)(dx+i\,dy)$$

$$= \int_{x=0}^{1} (x^2-y^2)dx - 2xy\,dy + i\int_{x=0}^{1} 2xy\,dx + (x^2-y^2)dy$$

$$= \int_{x=0}^{1} (x^2 - x^4)dx - 4x^4 dx + i\int_{x=0}^{1} (x^2 - 2x^5)dx + 2x^3 dx$$

$$= -\frac{2}{3} + i\frac{2}{3}$$

5 Evaluate $\int_{1-i}^{2+i} (2x+1+iy)dz$ along $(1-i)$ to $(2+i)$

Sol: Along $(1-i)$ to $(2+i)$ is the straight line AB joining the points $(1,-1)$ to $(2,1)$.

The equation of AB is given by

$$y - y_1 = \underbrace{\left(\frac{y_2 - y_1}{x_2 - x_1}\right)}_{slope} (x - x_1)$$

$$\Rightarrow y + 1 = \frac{1+1}{2-1}(x-1) = 2x - 2$$

$$\Rightarrow y = 2x - 3 \Rightarrow dy = 2dx$$

Also x varies from 1 to 2

$$\therefore \int_{1-i}^{2+i} (2x+1+iy)dz = \int_{x=1}^{2} (2x+1)dx - ydy + i\int_{x=1}^{2} ydx + (2x+1)dy$$

$$= \int_{x=1}^{2} (-2x+7)dx + i\int_{x=1}^{2} (6x-1)dx$$

$$= 4 + 8i.$$

6 Evaluate $\int_C \bar{z}dz$, $z = 0$ to $4+2i$ along C given by

(i) $t^2 + it$ (ii) Along the line $z = 0$ to 2 and there from 2 to $4+2i$.

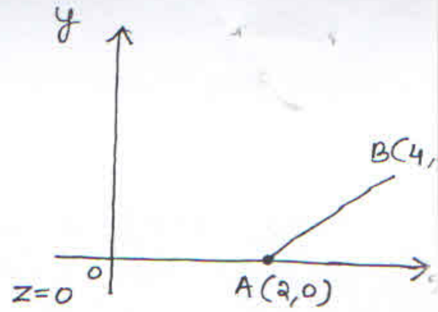
Sol: (i) Along $z(t) = t^2 + it \Rightarrow x = t^2$, $y = t$, t varies from 0 to 2

$$\Rightarrow dx = 2tdt \Rightarrow dy = dt.$$

$$\therefore \int_C (x - iy)(dx + idy) = 10 - \frac{8i}{3}.$$

(ii) Along OA :- $y=0$, $dy=0$

x varies from 0 to 2



$$\therefore \int_C \bar{z}\,dz = \int_{x=0}^{2} x\,dx = 2$$

Along AB :- Eqn of AB is $y = x-2$, $dy = dx$ | $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$

x varies from 2 to 4 .

$$\therefore \int_C \bar{z}\,dz = \int_{x=0}^{2} (x-iy)(dx+idy) = 8+4i$$

Thus, $\displaystyle\int_C \bar{z}\,dz = 2+8+4i = \underline{\underline{10+4i}}$

$\boxed{7}$ Evaluate $\displaystyle\int_C \bar{z}^2 dz$ along (i) C is $|z|=1$

(ii) C is $|z-1|=1$

Sol: (i) Given $C : |z| = 1$ $\left[\,|z-0|=1\,\right]$

$$\Rightarrow z = e^{i\theta}, \quad dz = ie^{i\theta}d\theta$$

$$\Rightarrow \bar{z} = e^{-i\theta}$$

$$\Rightarrow \bar{z}^2 = e^{-2i\theta}$$

Also θ varies from 0 to 2π

NOTE:

$z-a = re^{i\theta}$

Eqn of cle centre a radius r.

$|z-a| = |r|\,|e^{i\theta}| = r(1)$

$|e^{i\theta}| = |\cos\theta + i\sin\theta|$

$= \sqrt{\cos^2\theta+\sin^2\theta} = \sqrt{1} = 1$

$$\therefore \int_C \bar{z}^2 dz = i\int_0^{2\pi} e^{-2i\theta} e^{i\theta}\,d\theta = i\int_0^{2\pi} e^{-i\theta} d\theta = 0 .$$

(ii) Given $C : |z-1| = 1$

| $z = a + re^{i\theta}$
| $z-a = re^{i\theta}$

$$\Rightarrow z-1 = e^{i\theta}$$

$$\Rightarrow z = 1+e^{i\theta}, \quad dz = ie^{i\theta}d\theta$$

$$\bar{z} = 1+e^{-i\theta}$$

Also θ varies from 0 to 2π

$$\therefore \int_C \bar{z}^2 dz = \int_0^{2\pi} i\,(1+e^{-i\theta})^2 e^{i\theta}\,d\theta$$

$$= i\int_{\theta=0}^{2\pi} (e^{i\theta}+2+e^{i\theta})\,d\theta .$$

$$= i\left[\frac{e^{i\theta}}{i} + 2\theta + \frac{e^{-i\theta}}{-i}\right]_0^{2\pi} = 4\pi i.$$

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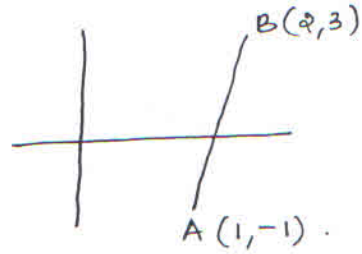
8 Evaluate $\int_{(0,0)}^{(1,1)} [3x^2+5y+i(x^2-y^2)]\,dz$ along $y^2 = x$.

Sol: Along $y^2 = x$, $2y\,dy = dx$, y varies from 0 to 1.

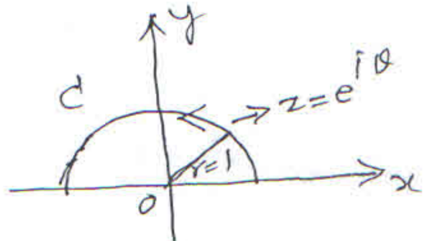
Ans: $\frac{129}{30} + \frac{44i}{15}$.

9 Evaluate $\int_{1-i}^{2+3i} (z^3+z)\,dz$ along the line joining $z = 1-i$ to $2+3i$.

Ans. $-\frac{125}{4} - 23i$ | $y = 4x-5$, $x \to 1$ to 2



10 Evaluate $\int_C (z-z^2)\,dz$, where C is the upper half of the circle $|z| = 1$



Sol: Given $|z| = 1$

$$\Rightarrow \quad z = e^{i\theta}$$
$$dz = i e^{i\theta}\,d\theta$$

Also θ varies from 0 to π (Semi circle)

$$\therefore \int_C (z-z^2)\,dz = \int_{\theta=0}^{\pi} \left[e^{i\theta} - (e^{i\theta})^2\right]\left(i e^{i\theta}\,d\theta\right)$$

$$= i\int_{\theta=0}^{\pi} \left[e^{2i\theta} - e^{3i\theta}\right] d\theta$$

$$= \cancel{i}\left[\frac{e^{2i\theta}}{2\cancel{i}} - \frac{e^{3i\theta}}{3\cancel{i}}\right]_0^{\pi} = \cancel{\frac{1}{2i}}\left[\frac{e^{i2\theta}}{2} - \frac{e^{i3\theta}}{3}\right]_0^{\pi} = \boxed{\frac{2}{3}}$$

Ans: $\boxed{\frac{2}{3}}$

$$\cancel{= \frac{1}{6}\left[3(e^{i2\pi} - e^{i3\pi}) - 2(e^0 - e^0)\right] = \frac{1}{6}\left[3\{(\cos 2\pi - i\sin 2\pi) - (\cos 3\pi - i\sin 3\pi)\}\right]}$$

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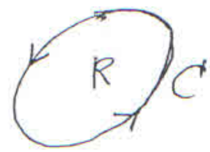
CAUCHY'S THEOREM

If a complex function $f(z)$ is analytic on and within a simple closed curve C, then $\int_C f(z)\,dz = 0$.

PROOF: Let $f(z) = u + iv$ is an Analytic function,



Then $$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \longrightarrow \boxed{1}$$

If R is the region bounded by C, we have by Greens theorem.

(i) $$\int_C u\,dx - v\,dy = \int_C u\,dx + (-v)\,dy = \iint_R \left[\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy$$

$$= -\iint_R \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dx\,dy \longrightarrow \boxed{2}$$

(ii) $$\int_C v\,dx + u\,dy = \iint_R \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dx\,dy \longrightarrow \boxed{3}$$

Substituting $\boxed{2}$ and $\boxed{3}$ in $\boxed{1}$, we get

$$\int_C f(z)\,dz = -\iint_R \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dx\,dy + i\iint_R \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dx\,dy$$

By CR equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\therefore \int_C f(z)\,dz = 0 + i(0) = 0$$

Hence the Theorem

NOTE:

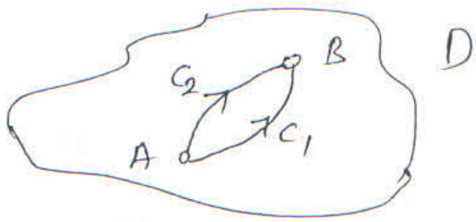
1. Greens Theorem: The Line integral $Li + Mj$ around C in a positively direction is equal to the integral over R of $\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$.

ie. $$\int_C L\,dx + M\,dy = \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy.$$

2. ~~Green~~ Cauchys theorem establishes one of the basic properties of an analytic function that the integral of an analytic function around any simple closed curve lying entirely in the simply connected domain of its analyticity is ZERO.

COROLLARIES:

1. Let $f(z)$ be analytic in a simply connected region D and A and B be any two points in D. Then the $I = \int_A^B f(z)\,dz$ does not depend on the parts of the integration joining A & B.



ie, I is same for all curves joining A & B.

2. Let C and C_1 be two positively oriented simple closed curves such that C_1 lies entirely within C. Let $f(z)$ be analytic on C and C_1 and in the region bounded by C and C_1

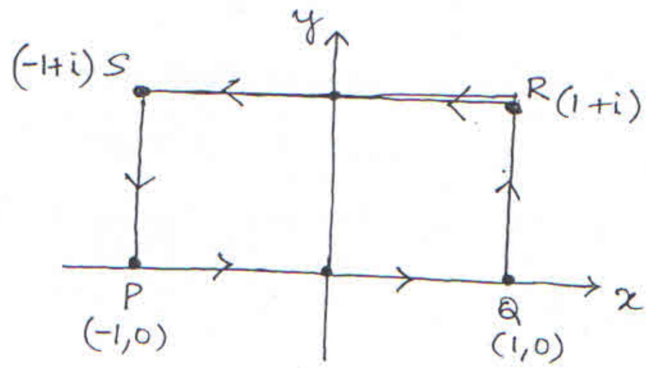
Then $$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz.$$

3. Let $C_1, C_2, \ldots, C_n$ be non-overlapping simple closed curves all of which lie within a simple closed curve C and $f(z)$ be analytic on these curves and in the region bounded by them, then
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

PROBLEMS

1. Verify Cauchy's Theorem for the integral of $f(z)=z^3$ taken along the boundary of the rectangle with vertices $(-1,0)$ $(1,0)$ $(1+i)$, ~~(-1)~~ $(-1+i)$.

Sol: The curve C is the rectangle in the complex plane



Since $f(z)=z^3$ is analytic everywhere in the complex plane, it is analytic on and within C. dy

By Cauchy's theorem, $\int_C f(z)dz=0$

Consider, $\oint_C z^3 dz = \int_{PQ} z^3 dz + \int_{QR} z^3 dz + \int_{RS} z^3 dz + \int_{SP} z^3 dz \longrightarrow (1)$

On PQ: $y=0$, $dy=0$ x varies from -1 to 1, $z=x$,

$$dz = dx + i dy = dx.$$

$$\therefore \int_{PQ} z^3 dz = \int_{x=-1}^{1} x^3 dx = 0.$$

On QR: $x=1$, $dx=0$, y varies from 0 to 1, $z=1+iy$, $dz=idy$

$$\therefore \int_{QR} z^3 dz = \int_{y=0}^{1} (1+iy)^3 (i dy) = \frac{1}{4}\left\{(1+i)^4 - 1\right\}$$

On RS: $y=1$, $dy=0$, x varies from 1 to -1, $z=x+i$,

$$dz = dx + i dy = dx$$

$$\therefore \int_{RS} z^3 dz = \int_{x=+1}^{-1} (x+i)^3 dx = \frac{1}{4}\left[(-1+i)^4 - (1+i)^4\right]$$

On SP : $x=-1$, $dx=0$, y varies from 1 to 0, $z=-1+iy$, $dz = i dy$

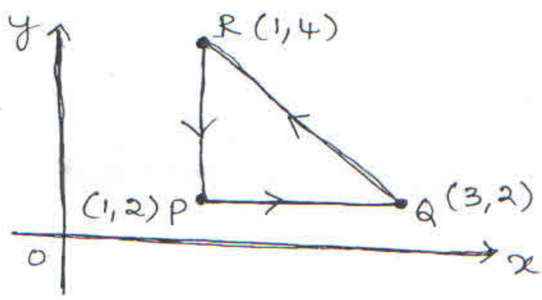
$$\therefore \int_{SP} z^3 dz = \int_1^0 (-1+iy)^3 (i dy) = \frac{1}{4}\left\{(-1)^4 - (-1+i)^4\right\}$$

$$\therefore \int_C z^3 dz = 0 + \frac{1}{4}\left\{(1+i)^4 - 1\right\} + \frac{1}{4}\left\{(-1+i)^4 - (1+i)^4\right\}$$
$$+ \frac{1}{4}\left\{1 - (-1+i)^4\right\} = 0.$$

Hence Cauchy's theorem is verified.

$\boxed{2}$ Verify Cauchy's theorem for the integral of $f(z) = \frac{1}{z}$ taken over the triangle formed by the points (1, 2) (3, 2) (1, 4)

Sol: The curve C is a triangle in the complex plane



St. line joining QR is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

Since $f(z) = \frac{1}{z}$ is analytic everywhere in the complex plane except at $z=0$, it is analytic on & within C.

$\therefore$ By Cauchy's theorem $\int_C f(z)\, dz = 0$

$$\text{Consider } \int_C \frac{1}{z} dz = \int_{PQ} \frac{1}{z} dz + \int_{QR} \frac{1}{z} dz + \int_{RP} \frac{1}{z} dz \longrightarrow \boxed{1}$$

<u>On PQ</u>: $y = 2$, $dy = 0$, $z = x + i2$, $dz = dx$, x varies from 1 to 3

$$\int_{PQ} \frac{1}{z}\,dz = \int_{x=1}^{3} \frac{dx}{x+2i} = \log(3+2i) - \log(1+2i)$$

<u>On QR</u>: $y = -x+5$, $dy = -dx$, x varies from 3 to 1, $dz = (1-i)d$

$z = x + i(-x+5)$.

$$\therefore \int_{QR} \frac{1}{z}\,dz = \int_{x=3}^{1} \frac{1}{(1-i)x+5i}(1-i)\,dx = (1-i)\int_{x=3}^{1} \frac{dx}{(1-i)x+5i}$$

$$= \cancel{(1-i)}\,\frac{\log[(1-i)x-5i]}{\cancel{(1-i)}}\Bigg]_3^1 = \log\big((1-i)+5i\big) - \log\big((1-i)3 + 5i\big)$$

$$= \log[1+4i] - \log[3+2i]$$

<u>On RP</u>: $x = 1$, $dx = 0$, y decreases from 4 to 2, $z = \cancel{x}1 + iy$, $dz =$

$$\int_{RP} \frac{1}{z}\,dz = \int_{y=4}^{2} \frac{i\,dy}{(1+iy)} = \log(1+iy)\Big]_4^2 = \log(1+2i) - \log(1+4i)$$

$$\therefore \int_C \frac{1}{z}\,dz = 0$$

Hence Cauchy's Theorem is verified.

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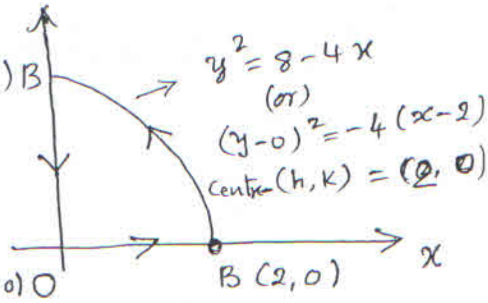
3. verify cauchy's Theorem for the integral of $f(z) = z$ taken over the curve C consisting of the +ve x-axis, the +ve y-axis and the parabola $y^2 = 8 - 4x$ in the +ve quadrant.

Sol: The curve C is given by the following figure.



The points of intersection are

$A = (2, 0)$, $B = (0, 2\sqrt{2})$, & $O = (0,0)$

Since $f(z) = z$ is analytic everywhere in the complexplane. it is analytic on and within C

$\therefore$ By Cauchy's Theorem $\int_C f(z)\,dz = 0$.

Consider $\int_C z\,dz = \int_{OA} z\,dz + \int_{AB} z\,dz + \int_{BO} z\,dz \longrightarrow \boxed{1}$

Along OA: $y = 0$, $dy = 0$, $x \to 0$ to 2

$$\therefore \int_{OA} z\,dz = \int_{x=0}^{2} (x+iy)(dx+idy) = \int_{x=0}^{2} x\,dx = \left.\frac{x^2}{2}\right]_0^2 = 2$$

Along AB: $y^2 = 8 - 4x \Rightarrow 2y\,dy = -4\,dx \Rightarrow dx = -\left(\frac{y}{2}\right)dy$

$y \to 0$ to $2\sqrt{2}$

$$\therefore \int_{AB} z\,dz = \int_{y=0}^{2\sqrt{2}} (x+iy)(dx+idy) = \int_{y=0}^{2\sqrt{2}} \left[\frac{8-y^2}{4} + iy\right]\left(\frac{8-2y}{4} + i\right)dy = -6$$

Along BO: $x = 0$, $dx = 0$, $y \to 2\sqrt{2}$ to 0

$$\therefore \int_{BO} z\,dz = \int_{y=2\sqrt{2}}^{0} (\cancel{x}+iy)(\cancel{dx}+idy) = \int_{y=2\sqrt{2}}^{0} i^2 y\,dy = -\int_{y=2\sqrt{2}}^{0} y\,dy = \int_{y=0}^{2\sqrt{2}} y\,dy = 4.$$

$\therefore \int_C z\,dz = 2 - 6 + 4 = 0$, Hence Cauchy Theorem is verified

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NOTE :

Proper Fraction : Degree of Nr < Degree of Dr.

Improper Fraction : Degree of Nr $\geq$ Degree of Dr.

Converting Improper fraction to Proper fraction.

Formula: $\underset{\text{Quotient}}{Q} + \dfrac{R \rightarrow \text{Remainder}}{D \rightarrow \text{Divider}}$

Standard nth derivaties:

$$\frac{d^n}{dx^n}\left[\frac{1}{ax+b}\right] = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$$

$$\frac{d^n}{dx^n}\left[\operatorname{Log}(ax+b)\right] = \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$$

$$\frac{d^n}{dx^n}\left[\sin(ax+b)\right] = a^n \sin\left[(ax+b) + \frac{n\pi}{2}\right]$$

$$\frac{d^n}{dx^n}\left[\cos(ax+b)\right] = a^n \cos\left[(ax+b) + \frac{n\pi}{2}\right]$$

Binomial Expansions:

$$\left.\begin{aligned}
(1+x)^{-1} &= 1 - x + x^2 - x^3 + \cdots \\
(1-x)^{-1} &= 1 + x + x^2 + x^3 + \cdots \\
(1+x)^{-2} &= 1 - 2x + 3x^2 - 4x^3 + \cdots \\
(1-x)^{-2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots
\end{aligned}\right\}$$

These are useful for Laurentz's series expansions.

These are valid for only when $|x| < 1$.

TAYLOR'S SERIES

Suppose a complex function $f(z)$ is analytic at all points on and within the circle $C : |z-a| = r$, Then at each point z which is Inside C, we have

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

(OR)

$$f(z) = f(a) + \sum_{n=1}^{\infty} \frac{f^n(a)}{n!} (z-a)^n$$

The RHS of the above expression is an Infinite series in Ascending powers of $(z-a)$. This series is called the Taylor Series for the function $f(z)$ about the point a

NOTE :

i) If $a=0$, Then $f(z) = f(0) + \sum_{n=1}^{\infty} \frac{f^n(0)}{n!} z^n$. which is called the Maclaurin's series for the function $f(z)$.

ii) Exponential series : $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{z^n}{n!}$

iii) Logarithmic series : $\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$

iv) Sine series : $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=1}^{\infty} \frac{z^n}{n!} \sin\left(\frac{n\pi}{2}\right)$

(v) Cosine series : $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

(vi) Binomial Series : $(1+z)^m = 1 + mz + \frac{m(m-1)z^2}{2!} + \frac{m(m-1)(m-2)}{3!} z^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} z^n.$$

PROBLEMS : (Taylor series)

[1] Expand $f(z) = \sin z$ in Taylor's series about the point $a = \frac{\pi}{4}$

Sol: By Taylor series we have

$$f(z) = f\left(\tfrac{\pi}{4}\right) + \sum_{n=1}^{\infty} \frac{f^{n}(\pi/4)}{n!}\left(z - \pi/4\right)^{n} \longrightarrow \text{I}$$

Here $f(z) = \sin z$, $a = \frac{\pi}{4}$ $\Rightarrow f(\pi/4) = \frac{1}{\sqrt{2}}$

$$f^{n}(z) = \sin\left(z + \frac{n\pi}{2}\right) \Rightarrow f^{n}(\pi/4) = \sin\left(\frac{\pi}{4} + \frac{n\pi}{2}\right) = \sin\frac{(2n+1)\pi}{4}$$

substituting in I we get

$$\boxed{\sin z = \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \frac{(z - \pi/4)^{n}}{n!} \sin\frac{(2n+1)\pi}{4}.}$$

[2] Expand $f(z) = \frac{z-1}{z+1}$ in Taylor series about
(i) the point $a = 0$ (ii) the point $a = 1$.

$\longrightarrow Q + \frac{R}{D}$

Divider

Sol: (i) Given $f(z) = \frac{z-1}{z+1} = 1 - \frac{2}{z+1}$ & $a = 0$.

$$\begin{array}{l} \overset{D}{\downarrow} \\ z+1 \overline{)\, z-1 \,}\, (1 \rightarrow Q \\ \quad\ \ \underline{(-)\, z+1} \\ \qquad\quad -2 \rightarrow R \end{array}$$

By Taylor series

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{z^{n}}{n!} f^{n}(0) \longrightarrow \text{II}$$

Here $f(z) = 1 - \frac{2}{z+1}$ $\Rightarrow f(0) = 1 - \frac{2}{1} = -\cancel{2}$

$$\& \ f^{n}(z) = -2\left[\frac{(-1)^{n} n!}{(z+1)^{n+1}}\right] \Rightarrow f^{n}(0) = -2\left[\frac{(-1)^{n} n!}{1^{n+1}}\right] = 2(-1)^{n+1} n!$$

Substituting in II we get

$$\frac{z-1}{z+1} = -1 + \sum_{n=1}^{\infty} \frac{2 z^{n}}{\cancel{n!}} (-1)^{n+1} (\cancel{n!}) = -1 + 2\sum_{n=1}^{\infty} (-1)^{n+1} z^{n}$$

(ii) Given $f(z) = \frac{z-1}{z+1} = 1 - \frac{2}{z+1}$ & $a = 1$.

By Taylor's series we have

$$f(z) = f(1) + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} f^n(1) \longrightarrow \boxed{1}$$

Here $f(z) = 1 - \frac{2}{z+1} \Rightarrow f(1) = 1 - \frac{2}{1+1} = 0$

$$f^n(z) = -2\left[\frac{(-1)^n n!}{(z+1)^{n+1}}\right] \Rightarrow f^n(1) = -2\left[\frac{(-1)^n n!}{2^{n+1}}\right]$$

$$f^n(z) = -\cancel{2}\frac{(-1)^n n!}{2^n \cdot \cancel{2}} = \frac{(-1)^{n+1} n!}{2^n}$$

Substituting in $\boxed{1}$ we get

$$\frac{z-1}{z+1} = 0 + \sum_{n=1}^{\infty} \frac{(z-1)^n}{\cancel{n!}} \frac{(-1)^{n+1}\cancel{n!}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}(z-1)^n$$

13 Find the Taylor series of $f(z) = \frac{1}{(z+1)^2}$ about the point $a = -i$

Sol: Given $f(z) = \frac{1}{(z+1)^2}$; $a = -i$

By Taylor series, we have

$$f(z) = f(-i) + \sum_{n=1}^{\infty} \frac{f^n(-i)}{n!}(z+i)^n \longrightarrow \boxed{1}$$

Here $f(z) = \frac{1}{(z+1)^2} \Rightarrow f(-i) = \frac{1}{(-i+1)^2} = \frac{1}{-2i} = \frac{i}{2}$

?

& $f^n(z) = \frac{(-1)^n (n+1)!}{(z+1)^{n+2}} \Rightarrow f^n(-i) = \frac{(-1)^n (n+1)!}{(-i+1)^{n+2}} = \frac{1}{(1-i)^2} \frac{(-1)^n (n+1)!}{(1-i)^n}$

$$= \frac{i}{2} \cdot \frac{(-1)^n (n+1)!}{(1-i)^n}$$

substituting in III we get

$$\frac{1}{(z+1)^2} = \frac{i}{2} + \sum_{n=1}^{\infty} \frac{(z+i)^n}{n!} \frac{(-1)^n (n+1)!}{(1-i)^n} \times \frac{i}{2}$$

$$\frac{1}{(z+1)^2} = \frac{i}{2} + \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{(1-i)^n} (z+i)^n \right\}$$

(4) Find the Taylor series of $f(z) = \frac{2z^3+1}{z^2+z}$ about the point $a = i$.

Sol: Given $f(z) = \frac{2z^3+1}{z^2+z}$; $a = i$

$$\begin{array}{r l}
z^2+z \,) & 2z^3+1 \,(2z-2 \\
(-) & 2z^3+2z^2 \\
\hline
& 1-2z^2 \\
(-) & -2z-2z \\
\hline
& 1+2z \rightarrow R
\end{array}$$

$\rightarrow D$, $\rightarrow Q$

By Taylor series, we have

$$f(z) = f(i) + \sum_{n=1}^{\infty} \frac{(z-i)^n}{n!} f^n(i) \longrightarrow \text{III}$$

iv

Here $f(z) = \frac{2z^3+1}{z^2+z} = (2z-2) + \frac{2z+1}{z^2+z} = 2(z-1) + \frac{(z+1)+z}{z(z+1)}$

$$= 2(z-1) + \frac{1}{z} + \frac{z}{z+1}$$

$$\Rightarrow f(i) = 2(i-1) + \frac{1}{i} + \frac{i}{i+1} = 2(i-1) - i + \frac{i(i-1)}{i^2-1}$$

$$= 2i-2-i + \frac{i^2-i}{-2} = i-2 - \frac{i}{-2} + \frac{i}{2} = \frac{i}{2} - \frac{3}{2}$$

$$f'(z) = 2 - \frac{1}{z^2} - \frac{1}{(z+1)^2} \Rightarrow f'(i) = 3 + \frac{i}{2}$$

$$f^{n}(z) = \frac{(-1)^n n!}{z^{n+1}} + \frac{(-1)^n n!}{(z+1)^{n+1}} \quad , n \geq 2$$

$$\Rightarrow f^{n}(i) = \frac{(-1)^n n!}{i^{n+1}} + \frac{(-1)^n n!}{(1+i)^{n+1}} = (-1)^n n! \left[\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}} \right] , n \geq 2$$

Substituting the above values in III we get

$$f(z) = \left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 + \frac{i}{2}\right)(z-i) + \sum_{n=2}^{\infty} (-1)^n (n!) \left\{ \frac{1}{i^{n+1}} + \frac{1}{(i+1)^{n+1}} \right\} (z-i)^n$$

LAURENT SERIES

- Let $f(z)$ be analytic on two concentric circles C_1 & C_2 with centre Z_0 and radii R_1 and R_2 and in the <u>annular region</u> $R_1 < |z - z_0| < R_2$, Then $f(z)$ is uniquely represented by a convergent series given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \longrightarrow \boxed{1}$$

This series is known as Laurent's Series of $f(z)$ about the point z_0

Where
$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\omega)}{(\omega - a)^{n+1}} d\omega, \quad n = 0, 1, 2, 3, \dots \boxed{2}$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\omega)}{(\omega - a)^{-n+1}} d\omega, \quad n = 1, 2, 3, \dots \boxed{3}$$

NOTE:

* The First part in the RHS of expression $\boxed{1}$ is called the ANALYTIC part of $f(z)$ and the second part is called the PRINCIPAL part of $f(z)$.

* We observe that the RHS of expression $\boxed{1}$ is a power series which contains Non-Negative powers of $(z-a)$.

* The Evaluation of the coefficients a_n and b_n by using Formula (2) and (3) is more complex in general. Usually a rational function is expanded by using Known Expansions Like Binomial, Exponential and Logarithmic, ETC.

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Problems on Laurent's Series

1. Expand the function $f(z) = \dfrac{z+1}{(z+2)(z+3)}$ in Laurent series valid for (i) $|z| < 2$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$.

Sol: Given $f(z) = \dfrac{z+1}{(z+2)(z+3)} = \dfrac{2}{z+3} - \dfrac{1}{z+2}$

$$\cancel{\frac{z+1}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}} \quad \cancel{\Rightarrow z+1 = A(z+3) + B(z+2)}$$

$$\frac{z+1}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3} \Rightarrow z+1 = A(z+3) + B(z+2)$$

$$z = -2, \quad -1 = A(1) \Rightarrow \boxed{A = -1}$$

$$z = -3, \quad -2 = B(-1) \Rightarrow \boxed{B = 2}$$

(i) Let $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$

$$\therefore f(z) = \frac{2}{3(1+z/3)} - \frac{1}{2(1+z/2)} = \frac{2}{3}\left(1+\frac{z}{3}\right)^{-1} - \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{2}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right] - \frac{1}{2}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

is the required series.

(ii) Let $2 < |z| < 3 \Rightarrow |2/z| < 1$ and $|z/3| < 1$

$$\therefore f(z) = \frac{2}{3(1+z/3)} - \frac{1}{z(1+2/z)}$$

$$= \frac{2}{3}\left(1+\frac{z}{3}\right)^{-1} - \frac{1}{z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= \frac{2}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right] - \frac{1}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \cdots\right]$$

is the required series.

(iii) Let $|z| > 3$ (or) $3 < |z| \Rightarrow \left|\frac{3}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$

$$\therefore f(z) = \frac{2}{z(1+3/z)} - \frac{1}{z(1+2/z)} = \frac{2}{z}\left(1+\frac{3}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{2}{z}\right)^{-1}$$

$$f(z) = \frac{2}{z}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \cdots\right] - \frac{1}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \cdots\right]$$

is the required series.

$\boxed{2}$ obtain the power series which represents the function $f(z) = \dfrac{z^2-1}{z^2+5z+6}$ in the following regions

(a) $|z| < 2$ (b) $2 < |z| < 3$ (c) $|z| > 3$

Sol: Given $f(z) = \dfrac{z^2-1}{z^2+5z+6} = 1 - \dfrac{5z+7}{z^2+5z+6}$

$$\begin{array}{l} \quad \nearrow D \\ z^2+5z+6)\ z^2-1\ (1 \leftarrow Q \\ \quad (-)\ z^2+5z+6 \\ \quad \quad -5z-7 \rightarrow R \end{array}$$

$$f(z) = 1 - \frac{5z+7}{(z+2)(z+3)}$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$\dfrac{5z+7}{(z+2)(z+3)} = \dfrac{A}{z+2} + \dfrac{B}{z+3}$

$\Rightarrow 5z+7 = A(z+3) + B(z+2)$

$z=-3,\ 8 = B(-1) \Rightarrow \boxed{B=-8}$

$z=-2,\ -3 = A(+1) \Rightarrow \boxed{A=-3}$

(a) Let $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$

$$\therefore f(z) = 1 + \frac{3}{2(1+\frac{z}{2})} - \frac{8}{3(1+z/3)} = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left\{1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right\} - \frac{8}{3}\left\{1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right\}$$

is the required series.

(ii) Let $2 < |z| < 3$ then $\left|\frac{2}{z}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$

$$\therefore f(z) = 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+z/3)} = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left\{1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right\} - \frac{8}{3}\left\{1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^2 + \cdots\right\}$$

is the required series.

(iii) Let $|z| > 3$ (or) $3 < |z| \Rightarrow \left|\frac{3}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$

$$\therefore f(z) = 1 + \frac{3}{z(1+2/z)} - \frac{8}{z(1+3/z)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left\{1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right\} - \frac{8}{z}\left\{1 - \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \cdots\right\}$$

is the required series.

$\boxed{3}$ Find the power series expansions of $f(z) = \frac{z}{(z^2+1)(z^2+4)}$ in the following regions (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$.

Sol: Given $f(z) = \frac{z}{(z^2+1)(z^2+4)} = \frac{z}{3}\left\{\frac{1}{z^2+1} - \frac{1}{z^2+4}\right\}$

$$\frac{z}{(z^2+1)(z^2+4)} = \frac{Az+B}{(z^2+1)} + \frac{Cz+D}{z^2+4}$$

$$\Rightarrow z = (Az+B)(z^2+4) + (Cz+D)(z^2+1)$$

comparing & simplying we get

(i) $|z|<1 \Rightarrow |z|^2<1 \Rightarrow \left|\frac{z}{1}\right|^2<1$ and $\left|\frac{z}{2}\right|^2<1$

$$\therefore f(z)=\frac{z}{3}\left\{\frac{1}{1+z^2}-\frac{1}{4\left(1+(z/2)^2\right)}\right\}$$

$$=\frac{z}{3}\left(1+z^2\right)^{-1}-\frac{1}{4}\left[1+\left(\frac{z}{2}\right)^2\right]^{-1}$$

$$=\frac{z}{3}\left[1-z^2+z^4-z^6+\cdots\right]-\frac{z}{12}\left[1-\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^4-\left(\frac{z}{2}\right)^6+\cdots\right]$$

is the required series.

(ii) Let $1<|z|<2 \Rightarrow \left|\frac{1}{z}\right|<1$ and $\left|\frac{1}{z}\right|^2<1$

Also $\left|\frac{z}{2}\right|<1$ and $\left|\frac{z}{2}\right|^2<1$

$$\therefore f(z)=\frac{z}{3}\left[\frac{1}{z^2\left(1+\frac{1}{z^2}\right)}-\frac{1}{4\left(1+(z/2)^2\right)}\right]$$

$$=\frac{\cancel{z}}{3}\left[\frac{1}{z}\left(1+\frac{1}{z^2}\right)^{-1}-\frac{z}{4}\left(1+\left(\frac{z}{2}\right)^2\right)^{-1}\right]$$

$$=\frac{1}{3}\left[\frac{1}{z}\left\{1-\frac{1}{z^2}+\left(\frac{1}{z^2}\right)^2-\left(\frac{1}{z^2}\right)^3+\cdots\right\}-\frac{z}{4}\left\{1-\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^4-\left(\frac{z}{2}\right)^6+\cdots\right\}\right]$$

is the required series.

(iii) Let $|z|>2$ then $\left|\frac{2}{z}\right|<1 \Rightarrow \left|\frac{2}{z}\right|^2<1$ and $\left|\frac{1}{z}\right|^2<1$

$$\therefore f(z)=\frac{3}{z}\left[\frac{1}{z^2(1+1/z^2)}-\frac{1}{z^2(1+4/z^2)}\right]$$

$$=\frac{3}{z}\left[\frac{1}{z^2}\left\{1+\frac{1}{z^2}\right\}^{-1}-\frac{1}{z^2}\left\{1+\frac{4}{z^2}\right\}^{-1}\right]$$

$$=\frac{3}{z^3}\left\{1-\frac{1}{z^2}+\left(\frac{1}{z^2}\right)^2-\cdots\right\}-\frac{3}{z^3}\left\{1-\frac{4}{z^2}+\left(\frac{4}{z^2}\right)^2-\cdots\right\}$$

is the required series

4 Expand the function $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for (a) $0 < |z+1| < 2$ (b) $|z+1| > 2$.

Sol: Given $f(z) = \frac{1}{(z+1)(z+3)}$

(a) Let $u = z+1$, Then $0 < |z+1| < 2 \Rightarrow |u| < 2$ (or) $\left|\frac{u}{2}\right| < 1$

$$\therefore f(z) = \frac{1}{u(u+2)} = \frac{1}{2u\left(1+\frac{u}{2}\right)} = \frac{1}{2u}\left[1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u}\left[1 - \left(\frac{u}{2}\right) + \left(\frac{u}{2}\right)^2 - \left(\frac{u}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2}\left[\frac{1}{u} - \frac{1}{2} + \frac{u}{2^2} - \frac{u^2}{2^3} + \cdots\right]$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{2} + \frac{z+1}{2^2} - \frac{(z+1)^2}{2^3} + \cdots\right]$$

is the required series.

(b) Let $|z+1| > 2 \Rightarrow |u| > 2$ (or) $\left|\frac{2}{u}\right| < 1$

$$\therefore f(z) = \frac{1}{u(u+2)} = \frac{1}{u^2\left(1+\frac{2}{u}\right)} = \frac{1}{u^2}\left(1+\frac{2}{u}\right)^{-1}$$

$$= \frac{1}{u^2}\left[1 - \frac{2}{u} + \frac{2^2}{u^2} - \frac{2^3}{u^3} + \cdots\right]$$

$$= \frac{1}{u^2} - \frac{2}{u^3} + \frac{2^2}{u^4} - \frac{2^3}{u^5} + \cdots$$

$$= \frac{1}{(z+1)^2} - \frac{2}{(z+1)^3} + \frac{2^2}{(z+1)^4} - \frac{2^3}{(z+1)^5} + \cdots$$

is the required Series.

1. Expand the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in power series in the following regions:

(a) $|z|<1$ (b) $1<|z|<2$ (c) $|z|>2$ (d) $0<|z-1|<1$ (e) $|z-1|>1$

Sol: Given $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

(a) Let $|z|<1$. Then $|z/2|<1$

$$\therefore f(z) = \frac{1}{1-z} - \frac{1}{2}\frac{1}{(1-z/2)} = (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \left\{1+z+z^2+z^3+\dots\right\} - \frac{1}{2}\left\{1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\dots\right\}$$

is the required series.

(b) Let $1<|z|<2$. Then $|1/z|<1$ and $|z/2|<1$

$$f(z) = -\frac{1}{2(1-z/2)} - \frac{1}{z(1-1/z)} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left\{1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\dots\right\} - \frac{1}{z}\left\{1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\dots\right\}$$

is the required series.

(c) Let $|z|>2$. Then $|2/z|<1$ and $|1/z|<1$

$$f(z) = \frac{1}{z(1-2/z)} - \frac{1}{z(1-1/z)} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left\{1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{3}{z}\right)^3+\dots\right\} - \frac{1}{z}\left\{1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\dots\right\}$$

is the required series.

(d) Let $0<|z-1|<1$. We now set $u=z-1$ so that $|u|<1$

$$\therefore f(z)=\frac{1}{u-1}-\frac{1}{u}=-(1-u)^{-1}-\frac{1}{u}=-\left\{1+u+u^2+u^3+\dots\right\}-\frac{1}{u}$$

$$=-\left\{1+(z-1)+(z-1)^2+(1-z)^3+\dots\right\}-\frac{1}{z-1}$$

is the required series.

(e) Let $|z-1|>1$. We set $u=z-1$. Then $|u|>1$ or $|1/u|<1$

$$f(z)=\frac{1}{u-1}-\frac{1}{u}=\frac{1}{u(1-1/u)}-\frac{1}{u}=\frac{1}{u}\left\{\left(1-\frac{1}{u}\right)^{-1}-1\right\}$$

$$=\frac{1}{u}\left[\left\{1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\left(\frac{1}{u}\right)^3+\dots\right\}-1\right]=\frac{1}{u^2}+\frac{1}{u^3}+\frac{1}{u^4}+\dots$$

$$=\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}+\dots$$

is the required series.

$\boxed{2}$ Obtain the Laurent series expansions of

$$f(z)=\frac{z^2}{(z-1)(z-3)}$$

in the following regions: (a) $1<|z|<3$ (b) $|z-1|<2$ (c) $|z-1|>2$

Sol: Given $f(z)=\dfrac{z^2}{(z-1)(z-3)}=1+\dfrac{4z-3}{(z-1)(z-3)}=1+\dfrac{1}{2}\cdot\dfrac{9}{z-3}-\dfrac{1}{2}\cdot\dfrac{1}{z-1}$

(a) $$f(z)=1+\frac{9}{2}\cdot\frac{1}{(-3)(1-z/3)}-\frac{1}{2}\cdot\frac{1}{z(1-1/z)}$$

$$=1-\frac{3}{2}\left(1-\frac{z}{3}\right)^{-1}-\frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1}$$

$$=1-\frac{3}{2}\left\{1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\dots\right\}-\frac{1}{2z}\left\{1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\dots\right\}$$

This is the Laurent series expansion of $f(z)$ for $1<|z|<3$.

(b) Let $|z-1|<2$. Let us set $u=z-1$. Then $|u/2|<1$

$$\therefore f(z) = 1+\frac{9}{2}\cdot\frac{1}{u-2} - \frac{1}{2}\cdot\frac{1}{u} = 1+\frac{9}{2}\cdot\frac{1}{(-2)(1-u/2)} - \frac{1}{2}\cdot\frac{1}{u}.$$

$$= 1-\frac{9}{4}\left(1-\frac{u}{2}\right) - \frac{1}{2u} = +1-\frac{9}{4}\left\{1+\frac{u}{2}+\left(\frac{u}{2}\right)^2+\left(\frac{u}{2}\right)^3+\cdots\right\} - \frac{1}{2u}$$

$$= 1-\frac{9}{4}\left\{1+\frac{1}{2}(z-1)+\frac{1}{2^2}(z-1)^2+\frac{1}{2^3}(z-1)^3+\cdots\right\} - \frac{1}{2}\cdot\frac{1}{z-1}$$

This is the Laurent series expansion of $f(z)$ for $|z-1|<2$.

(c) Let $|z-1|>2$. Then, if we set $u=z-1$, then $|u|>2$, or $|2/u|<1$,

$$\therefore f(z) = 1+\frac{9}{2}\cdot\frac{1}{u-2} - \frac{1}{2u} = 1+\frac{9}{2u}\cdot\frac{1}{(1-2/u)} - \frac{1}{2u}$$

$$= 1+\frac{1}{2u}\left\{9\left(1-\frac{2}{u}\right)^{-1}-1\right\}$$

$$= 1+\frac{1}{2u}\left[9\left\{1+\left(\frac{2}{u}\right)+\left(\frac{2}{u}\right)^2+\left(\frac{2}{u}\right)^3+\cdots\right\}-1\right].$$

$$= 1+\frac{4}{u}+9\left\{\frac{1}{u^2}+\frac{2}{u^3}+\frac{2^2}{u^4}+\cdots\right\}$$

$$= 1+\frac{4}{z-1}+9\left\{\frac{1}{(z-1)^2}+\frac{2}{(z-1)^3}+\frac{2^2}{(z-1)^4}+\cdots\right\}$$

This is the Laurent series expansion of $f(z)$ for $|z-1|>2$.

$\boxed{3}$ Expand the function, $f(z) = \dfrac{z}{z^2+1}$

in power series in the region $|z-3|>2$.

Sol: Let us set $z-3=u$. Then $|u|>2$, so that $\frac{1}{|u|}<\frac{1}{2}<1$. Then $\frac{1}{|u+3|}<1$ and $\frac{1}{|u+3|^2}<1$.

$$\therefore\ f(z) = \frac{z}{z^2+1} = \frac{1}{z(1+1/z^2)} = \frac{1}{z}\left(1+\frac{1}{z^2}\right)^{-1} = \frac{1}{(u+3)}\left[1+\frac{1}{(u+3)^2}\right]^{-1}$$

$$= \frac{1}{(u+3)}\left[1 - \frac{1}{(u+3)^2} + \frac{1}{(u+3)^4} - \frac{1}{(u+3)^6} + \cdots\right],$$

$$= \frac{1}{z}\left\{1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right\} = \frac{1}{z} - \frac{1}{z^3} + \frac{1}{z^5} - \frac{1}{z^7} + \cdots$$

This is the power series expansion of $f(z)$ for $|z-3| > 2$.

$\boxed{4}$ Expand the function

$$f(z) = \frac{z-1}{(z-2)(z-3)^2}$$

in Laurent series in the cases: (a) $|z| < 2$ (b) $2 < |z| < 3$ (c) $|z| > 3$.

Sol: Given $f(z) = \dfrac{z-1}{(z-2)(z-3)^2} = \dfrac{1}{z-2} - \dfrac{1}{z-3} + \dfrac{2}{(z-3)^2}$

(a) Let $|z| < 2$. Then $|z/2| < 1$ and $|z/3| < 1$

$$\therefore\ f(z) = \frac{1}{-2(1-z/2)} - \frac{1}{(-3)(1-z/3)} + \frac{2}{9(1-z/3)^2}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + \frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{2}{9}\left(1-\frac{2}{3}\right)^{-2}$$

$$= -\frac{1}{2}\left\{1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right\} + \frac{1}{3}\left\{1+\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right\}$$

$$+ \frac{2}{9}\left\{1+2\left(\frac{z}{3}\right)+3\left(\frac{z}{3}\right)^2+4\left(\frac{z}{3}\right)^3+\cdots\right\},$$

This is the Laurent series expansion of $f(z)$ for $|z| < 2$.

(b) Let $2 < |z| < 3$. Then $|2/z| < 1$ and $|z/3| < 1$.

$$\therefore\ f(z) = \frac{1}{z(1-2/z)} - \frac{1}{(-3)(1-z/3)} + \frac{2}{9(1-z/3)^2}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} + \frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{2}{9}\left(1-\frac{z}{3}\right)^{-2}$$

$$= \frac{1}{z}\left\{1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right\} + \frac{1}{3}\left\{1+\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right\}$$

$$+ \frac{2}{9}\left\{1+2\left(\frac{z}{3}\right)+3\left(\frac{z}{3}\right)^2+4\left(\frac{z}{3}\right)^3+\cdots\right\}.$$

This is the Laurent series expansion of $f(z)$ for $2<|z|<3$.

(c) Let $|z|>3$. Then $|3/z|<1$ and $|2/z|<1$.

$$\therefore f(z) = \frac{1}{z(1-2/z)} - \frac{1}{z(1-3/z)} + \frac{2}{z^2(1-3/z)^2}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} + \frac{2}{z^2}\left(1-\frac{3}{z}\right)^{-2}$$

$$= \frac{1}{z}\left\{1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right\} - \frac{1}{z}\left\{1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots\right\}$$

$$+ \frac{2}{z^2}\left\{1+2\left(\frac{3}{z}\right)+3\left(\frac{3}{z}\right)^2+4\left(\frac{3}{z}\right)^3+\cdots\right\}.$$

This is the Laurent series expansion of $f(z)$ for $|z|>3$.

$\boxed{5}$ Expand $f(z) = \dfrac{2z^2-3z+4}{(z-1)(z+2)^2}$ as Laurent series valid for

(a) $1<|z|<2$ and (b) $|z+1|>2$.

Sol:- Given, $f(z) = \frac{1}{3}\cdot\frac{1}{z-1} + \frac{5}{3}\cdot\frac{1}{z+2} - \frac{6}{(z+2)^2}$

(a) Let $1<|z|<2$. Then $|1/z|<1$ and $|z/2|<1$,

$$\therefore f(z) = \frac{1}{3}\cdot\frac{1}{z(1-1/z)} + \frac{5}{3}\cdot\frac{1}{2(1+z/2)} - \frac{6}{4(1+z/2)^2}$$

$$= \frac{1}{3z}\left(1-\frac{1}{z}\right)^{-1} + \frac{5}{6}\left(1+\frac{z}{2}\right)^{-1} - \frac{3}{2}\left(1+\frac{z}{2}\right)^{-2}$$

$$= \frac{1}{3z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{5}{6}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n - \frac{3}{2}\sum_{n=0}^{\infty}(n+1)\left(-\frac{z}{2}\right)^n$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \frac{5}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}z^n - \frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)(n+1)}{2^n}z^n$$

This is the Laurent series expansion of $f(z)$ for $1<|z|<2$.

(b) Let $|z+1|>2$. Let us set $u=z+1$ so that $|u|>2$, or $|2/u|<1$ and $|1/u|<1$.

$$\therefore f(z) = \frac{1}{3}\cdot\frac{1}{u-2} + \frac{5}{3}\cdot\frac{1}{u+1} - \frac{6}{(u+1)^2}$$

$$= \frac{1}{3}\cdot\frac{1}{u(1-2/u)} + \frac{5}{3}\cdot\frac{1}{u(1+1/u)} - \frac{6}{u^2(1+1/u)^2}$$

$$= \frac{1}{3u}\left(1-\frac{2}{u}\right)^{-1} + \frac{5}{3u}\left(1+\frac{1}{u}\right)^{-1} - \frac{6}{u^2}\left(1+\frac{1}{u}\right)^{-2}$$

$$= \frac{1}{3u}\sum_{n=0}^{\infty}\left(\frac{2}{u}\right)^n + \frac{5}{3u}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{u}\right)^n - \frac{6}{u^2}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{1}{u}\right)^n$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\frac{2^n}{u^{n+1}} + \frac{5}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{u^{n+1}} - 6\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{u^{n+2}}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\frac{2^n}{(z+1)^{n+1}} + \frac{5}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(z+1)^{n+1}} - 6\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(z+1)^{n+2}}$$

This is the Laurent series expansion of $f(z)$ for $|z+1|>2$

$\boxed{6}$ Expand $f(z) = \dfrac{z}{(z+1)(z+2)}$ in Laurent's series about the point $a=-2$.

Sol :- Let $z-a = z+2$, Let $u = z+2$

$$\therefore f(z) = \frac{u-2}{(u-1)u} = \frac{2-u}{u}(1-u)^{-1} = \left(\frac{2}{u}-1\right)\left\{1+u+u^2+\dots\right\} \text{ for } |u|<1$$

$$= 2\left\{\frac{1}{u}+1+u+u^2+\dots\right\} - \left\{1+u+u^2+\dots\right\}$$

$$= \frac{2}{u} + 1 + u + u^2 + \cdots$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots \quad \text{for } |z+2| < 1 .$$

This is the required expansion.

$\boxed{7}$ Expand $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ in Laurent's series about the point $a = 1$.

Sol:- Let $(z-a) = (z-1)$. Setting $u = z-1$.

$$\therefore f(z) = \frac{e^{2(u+1)}}{u^3} = \frac{e^2}{u^3} e^{2u} .$$

$$\Rightarrow f(z) = \frac{e^2}{u^3} \sum_{n=0}^{\infty} \frac{(2u)^n}{n!} = e^2 \sum_{n=0}^{\infty} \frac{2^n u^{n-3}}{n!}$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n (z-1)^{n-3}}{n!}$$

This is the required expansion.

SINGULAR POINTS :

A Point 'a' at which a complex function $f(z)$ fails to be analytic is called a singular point (or) Singularity of $f(z)$.

EX:- $z=0$ is a singular point for $f(z)=\log z$ and $f(z)=\frac{1}{z}$

$z=\frac{\pi}{2}$ is a singular point for $f(z)=\tan z$.

ISOLATED SINGULAR POINT :

A singular point 'a' of $f(z)$ is said to be an Isolated Singular point If there exists a Neighbourhood of 'a' which contains <u>No</u> other singular point of $f(z)$.

EX:- $z=0$ is a Isolated singular point of $f(z)=\frac{1}{z}$

($\because$ $f(z)=\frac{1}{z}$ is analytic every where except at $z=0$).

$z=1, 2i$ are the Isolated singular points of

$$f(z)=\frac{2z}{(z-1)(z-2i)}.$$

POLE :

Consider a Laurents Expansion of $f(z)$ about 'a'

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} \longrightarrow \boxed{1}$$

Suppose in the series, the principal part of $\boxed{1}$ terminates at $n=m$, where $m \geq 1$, So that $b_{m+1}=b_{m+2}=\cdots=0$, Then

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n + \frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+\cdots+\frac{b_m}{(z-a)^m}$$

In this case the point $z=a$ is called a pole of order m of $f(z)$.

In particular, Pole of order one is called simple pole
· pole of order two is called double pole
Pole of order three is called Triple pole.

RESIDUE :

Suppose a point 'a' is a pole of order 'm' of $f(z)$, then the coefficient of $\frac{1}{z-a}$ in the Laurents expansion of $f(z)$ about 'a' is called the Residue of $f(z)$ at the pole 'a'.

DETERMINATION OF POLES:

It can be proved that If $f(z) = \frac{\phi(z)}{(z-a)^m}$, where $\phi(z)$ is analytic and <u>not zero</u> at the point 'a', Then 'a' is a pole of order m of $f(z)$. In this case the poles of $f(z)$ may be obtained by solving the equation $\frac{1}{f(z)} = 0$.

EX: (i) $f(z) = \frac{1}{\cos z - \sin z} \Rightarrow \frac{1}{f(z)} = 0 \Rightarrow \cos z - \sin z = 0$
$\Rightarrow z = \frac{\pi}{4}$ is a simple pole of $f(z)$.

(ii) $f(z) = \frac{\cos z}{(z-1)^3} \Rightarrow \frac{1}{f(z)} = 0 \Rightarrow (z-1)^3 = 0$
$\Rightarrow z = 1$ is a Triple pole of $f(z)$.

(iii) $f(z) = \frac{z^2}{(z+1)^2(z^2+1)} \Rightarrow \frac{1}{f(z)} = 0 \Rightarrow$ $z=-1$ is a Double pole
$z = \pm i$ are simple poles.

DETERMINATION OF A RESIDUE

If 'a' is a pole of order $m \geq 1$ of $f(z)$ then the residue of $f(z)$ at 'a' is given by the formula.

$$\text{Residue of } f(z) \text{ at } a = \frac{1}{(m-1)!}\left[\lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}\right]$$

Ex:-

If $m=1$ (simple pole), Residue $= \lim_{z \to a} (z-a) f(z)$.

If $m=2$ (double pole), Residue $= \frac{1}{1!} \lim_{z \to a} \frac{d}{dz}\left\{(z-a)^2 f(z)\right\}$

If $m=3$ (Triple pole), Residue $= \frac{1}{2!} \lim_{z \to a} \frac{d^2}{dz^2}\left\{(z-a)^3 f(z)\right\}$

PROBLEMS ON POLES & RESIDUES

I. For the following functions, find the poles and residues at each pole.

(i) $f(z) = \dfrac{z^2+1}{z^2-2z}$

Sol: Here $f(z) = \dfrac{z^2+1}{z^2-2z} = \dfrac{z^2+1}{z(z-2)}$

To find Pole:

Consider $\dfrac{1}{f(z)} = 0 \Rightarrow \dfrac{z(z-2)}{z^2+1} = 0$

$\Rightarrow z(z-2) = 0$

$\Rightarrow z = 0$, & $z = 2$ are simple poles.

To Find Residue:

By Formula

$$\text{Residue at } 0 = \lim_{z\to 0} (z-0)\, f(z) = \lim_{z\to 0}\left[\cancel{z}\cdot\frac{(z^2+1)}{\cancel{z}(z-2)}\right] = \boxed{-\frac{1}{2}}$$

$$\text{Residue at } 2 = \lim_{z\to 2} (z-2)\, f(z) = \lim_{z\to 2}\left[\cancel{(z-2)}\,\frac{z^2+1}{z\cancel{(z-2)}}\right] = \frac{2^2+1}{2} = \boxed{\frac{5}{2}}$$

(ii) $f(z) = \dfrac{2z+1}{z^2-z-2}$

Sol: Here $f(z) = \dfrac{2z+1}{z^2-z-2} = \dfrac{2z+1}{(z+1)(z-2)}$

To find pole:

Consider $\dfrac{1}{f(z)} = 0 \Rightarrow (z+1)(z-2) = 0$

$\Rightarrow z = -1$, & $z = 2$ are simple poles.

To find Residue:

By Formula.

$$\text{Residue at } -1 = \lim_{z\to -1} (z+1)\, f(z) = \lim_{z\to -1} \frac{(2z+1)\cancel{(z+1)}}{\cancel{(z+1)}(z-2)} = \boxed{\frac{1}{3}}$$

Residue at 2 $= \lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} \cancel{(z-2)} \times \frac{2z+1}{\cancel{(z-2)}(z+1)}$

$$= \lim_{z\to 2} \frac{2z+1}{z+1} = \boxed{\frac{5}{3}}$$

(iii) $f(z) = \frac{z}{(z-1)(z^2+1)}$

Soln: Here $f(z) = \frac{z}{(z-1)(z^2+1)} = \frac{z}{(z-1)(z+i)(z-i)}$

To find pole :

consider $\frac{1}{f(z)} = 0 \Rightarrow \frac{(z-1)(z^2+1)}{z} = 0$

$$\Rightarrow (z-1)(z^2+1) = 0$$

$$\Rightarrow (z-1)(z+i)(z-i) = 0$$

$\Rightarrow \boxed{z = 1, -i, i}$ are the simple poles

To find Residue:

Residue at 1 $= \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \cancel{(z-1)} \cdot \frac{z}{\cancel{(z-1)}(z^2+1)} = \lim_{z\to 1} \frac{z}{z^2+1} = \boxed{\frac{1}{2}}$

Residue at $(-i) = \lim_{z\to -i} (z+i) f(z) = \lim_{z\to -i} \cancel{(z+i)} \cdot \frac{z}{(z-1)\cancel{(z+i)}(z-i)}$

$$= \lim_{z\to -i} \frac{z}{(z-1)(z-i)} = \frac{-\cancel{i}}{(-i-1)(-2\cancel{i})}$$

$$= \frac{1}{-2(1+i)} \times \frac{1-i}{1-i} = \frac{1-i}{-2(1^2-i^2)} = -\frac{1}{4}(1-i)$$

Residue at $(i) = \lim_{z\to i} (z-i) f(z) = \lim_{z\to i} \cancel{(z-i)} \frac{z}{(z-1)(z+i)\cancel{(z-i)}}$

$$= \lim_{z\to i} \frac{z}{(z-1)(z+i)} = \frac{\cancel{i}}{(i-1)(2\cancel{i})}$$

$$= \frac{1}{2(i-1)} \times \frac{i+1}{i+1} = \frac{i+1}{2(i^2-1)} = -\frac{1}{2}(1+i)$$

(iv) $f(z) = \dfrac{z^2+4}{z^3+2z^2+2z} = \dfrac{z^2+4}{z(z^2+2z+2)}$

<u>To find pole :</u>

Consider $\dfrac{1}{f(z)} = 0 \Rightarrow \dfrac{z(z^2+2z+2)}{z^2+4} = 0$

$ax^2+bx+c=0$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\Rightarrow z(\underbrace{z^2}_{a=1}+\underbrace{2z}_{b=2,}+\underbrace{2}_{c=2}) = 0$

$\Rightarrow z = 0, \quad z = \dfrac{-2 \pm \sqrt{2^2-4(1)(2)}}{2\times1}$

$= \dfrac{-2 \pm \sqrt{-4}}{2}$

$= \dfrac{-2 \pm \sqrt{4}\,i}{2}$ $\qquad \sqrt{-4} = \sqrt{4}.\sqrt{-1} = \underline{\underline{2}}\,i$

$= -1 \pm i$

$\therefore z = 0, \underbrace{(-1+i)}_{\alpha}, \underbrace{(-1-i)}_{\beta}$ are the simple poles.

<u>Residue at 0</u> $= \lim\limits_{z\to0}(z-0)f(z) = \lim\limits_{z\to0} \cancel{z}.\dfrac{z^2+4}{\cancel{z}(z^2+2z+2)} = \dfrac{4}{2} = \boxed{2}$

Residue at α $= \lim\limits_{z\to\alpha}(z-\alpha).f(z) = \lim\limits_{z\to\alpha}\cancel{(z-\alpha)}\dfrac{z^2+4}{z\cancel{(z-\alpha)}(z-\beta)}$

$(-1+i)$

$= \lim\limits_{z\to\alpha}\dfrac{z^2+4}{z(z-\beta)}$

$= \dfrac{\alpha^2+4}{\alpha(\alpha-\beta)} = \dfrac{(-1+i)^2+4}{(-1+i)(2i)}$

$\therefore$ Residue at $(-1+i)$ $= \dfrac{-2i+4}{-2i-2} = \dfrac{i-2}{i+1}\times\dfrac{i-1}{i-1} = \dfrac{3i-2}{2}.$

Residue at β $= \lim\limits_{z\to\beta}(z-\beta)f(z) = \lim\limits_{z\to\beta}\cancel{(z-\beta)}.\dfrac{z^2+4}{z(z-\alpha)\cancel{(z-\beta)}}$

$(-1-i)$

$= \lim\limits_{z\to\beta}\dfrac{z^2+4}{z(z-\alpha)}$

$= \dfrac{\beta^2+4}{\beta(\beta-\alpha)} = \dfrac{(-1-i)^2+4}{(-1-i)(-1-i+1-i)}$

$= \dfrac{-3i+1}{9}.$

(v) $f(z) = \dfrac{z^2}{z^4-1} = \dfrac{z^2}{(z^2-1)(z^2+1)}$

~~Sol~~ To find pole:

consider $\dfrac{1}{f(z)} = 0 \Rightarrow \dfrac{(z^2-1)(z^2+1)}{z^2} = 0$

$\Rightarrow (z^2-1)(z^2+1) = 0$

$z^2+1=0$
$z^2=-1$
$z = \pm\sqrt{-1} = \pm i$

$z^2-1=0$
$z^2 = 1$
$z = \pm\sqrt{1} = +1, -1.$

$\Rightarrow \boxed{z = -1, 1, -i, i}$ are the simple poles.

Residues:

Residue at $(-1) = \lim\limits_{z\to -1} (z+1) f(z) = \lim\limits_{z\to -1} (z+1)\dfrac{z^2}{(z^2-1)(z^2+1)} = \lim\limits_{z\to -1} \cancel{(z+1)}\dfrac{z^2}{\cancel{(z+1)}(z-1)(z^2+1)}$

$= \dfrac{(-1)^2}{(-1-1)[(-1)^2+1]} = \boxed{-\dfrac{1}{4}}$

Residue at $(1) = \lim\limits_{z\to 1}(z-1) f(z) = \lim\limits_{z\to 1} (z-1)\dfrac{z^2}{(z^2-1)(z^2+1)} = \lim\limits_{z\to 1} \cancel{(z-1)}\dfrac{z^2}{(z+1)\cancel{(z-1)}(z^2+1)}$

$= \lim\limits_{z\to 1} \dfrac{z^2}{(z+1)(z^2+1)} = \boxed{\dfrac{1}{4}}$

Residue at $(i) = \lim\limits_{z\to i}(z-i) f(z) = \lim\limits_{z\to i}(z-i)\dfrac{z^2}{(z^2-1)(z^2+1)} = \lim\limits_{z\to i}\cancel{(z-i)}\dfrac{z^2}{(z^2-1)(z+i)\cancel{(z-i)}}$

$= \lim\limits_{z\to i}\dfrac{z^2}{(z^2-1)(z+i)} = \dfrac{i^2}{(i^2-1)(i+i)} = \dfrac{-1}{(-2)(2i)}$

$= \dfrac{1}{4}\times\dfrac{1}{i}\times\dfrac{i}{i} = \boxed{-\dfrac{i}{4}}$

Residue at $(-i) = \lim\limits_{z\to -i}(z+i) f(z) = \lim\limits_{z\to -i}(z+i)\cdot\dfrac{z^2}{(z^2-1)(z^2+1)}$

$= \lim\limits_{z\to -i}\cancel{(z+i)}\cdot\dfrac{z^2}{(z^2-1)\cancel{(z+i)}(z-i)}$

$= \dfrac{(-i)^2}{[(-i)^2-1](-i-i)} = \dfrac{-1}{(-2)(-2i)} = -\dfrac{1}{4}\times\dfrac{1}{i}$

$= -\dfrac{1}{4}\times\dfrac{1}{i}\times\dfrac{i}{i} = \boxed{\dfrac{i}{4}}$

[2] For the following functions, Find the Poles and the Residue at each pole.

(i) $f(z) = \dfrac{z}{(z+1)(z-2)^2}$

Sol: <u>To find pole:</u>

Consider $\dfrac{1}{f(z)} = 0 \Rightarrow \dfrac{(z+1)(z-2)^2}{z} = 0 \Rightarrow (z+1)(z-2)^2 = 0$

$\Rightarrow z = -1$ is a simple pole

& $z = 2, 2$ Double pole $(m=2)$

<u>Residue:</u>

Residue at $(-1) = \lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} \cancel{(z+1)} \cdot \dfrac{z}{\cancel{(z+1)}(z-2)^2}$

$= \lim_{z \to -1} \dfrac{z}{(z-2)^2} = \boxed{\dfrac{-1}{9}}$

~~Residue at (2) = Lim (z-2) f(z)~~

Residue at 2 $(m=2)$ $= \dfrac{1}{1!} \lim_{z \to 2} \left[\dfrac{d}{dz} \left\{ (z-2)^2 \cdot f(z) \right\} \right]$

$= \lim_{z \to 2} \dfrac{d}{dz} \left[\cancel{(z-2)^2} \dfrac{z}{(z+1)\cancel{(z-2)^2}} \right]$

$= \lim_{z \to 2} \dfrac{d}{dz} \left[\dfrac{z}{z+1} \right]$

$= \lim_{z \to 2} \left[\dfrac{(z+1)(1) - z(1)}{(z+1)^2} \right]$

$= \lim_{z \to 2} \dfrac{1}{(z+1)^2} = \boxed{\dfrac{1}{9}}$

(ii) $f(z) = \dfrac{z^2}{(z-1)^2 (z+2)}$

<u>POLE:</u> Consider $\dfrac{1}{f(z)} = 0 \Rightarrow (z-1)^2 (z+2) = 0$

$\Rightarrow z = 1, 1$, Double pole

& $z = -2$. Simple pole

Residue:

$$\text{Residue at } (1) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz}\left[(z-1)^2 \cdot f(z)\right] = \lim_{z\to 1} \frac{d}{dz}\left[\cancel{(z-1)^2} \cdot \frac{z^2}{\cancel{(z-1)^2}(z+2)}\right]$$
$(m=2)$

$$= \lim_{z \to 1} \frac{d}{dz}\left[\frac{z^2}{(z+2)}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z+2)(2z) - z^2(1)}{(z+2)^2}\right]$$

$$= \frac{(1+2)(2(1)) - 1^2}{(1+2)^2} = \frac{6-1}{9} = \boxed{\frac{5}{9}}$$

$$\text{Residue at } (-2) = \lim_{z\to -2}\left[(z+2) f(z)\right]$$
$(m=1)$

$$= \lim_{z\to -2}\left[\cancel{(z+2)} \cdot \frac{z^2}{(z-1)^2 \cancel{(z+2)}}\right]$$

$$= \lim_{z\to -2}\left[\frac{z^2}{(z-1)^2}\right] = \frac{(-2)^2}{(-2-1)^2} = \boxed{\frac{4}{9}}$$

(iii) $$f(z) = \frac{z}{(z+1)^2(z^2+1)}$$

POLE:

Consider $\frac{1}{f(z)} = 0 \Rightarrow \frac{(z+1)^2(z^2+1)}{z} = 0$

$\Rightarrow (z+1)^2(z^2+1) = 0$

$z^2+1=0$
$\Rightarrow z^2 = -1$
$z = \pm\sqrt{-1} = \pm i$

$\Rightarrow z = -1, -1$ → Double pole

& $z = +i, -i$ → Simple pole

→

RESIDUE:

Residue at (-1) $(m=2)$ $= \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz}\left[\cancel{(z+1)^2} \cdot \frac{z}{\cancel{(z+1)^2}(z^2+1)}\right]$

$$= \lim_{z \to -1} \frac{d}{dz}\left[\frac{z}{z^2+1}\right] = \lim_{z \to -1}\left[\frac{(z^2+1)(1) - z(2z)}{(z^2+1)^2}\right]$$

$$= \frac{2-2}{(1+1)^2} = \frac{0}{4} = 0.$$

Residue at (i) $(m=1)$ $= \lim_{z \to i} (z-i) \cdot f(z) = \lim_{z \to i}\left[\cancel{(z-i)} \cdot \frac{z}{(z+1)^2(z+i)\cancel{(z-i)}}\right]$

$$= \lim_{z \to i}\left[\frac{z}{(z+1)^2(z+i)}\right] = \frac{\cancel{i}}{(i+1)^2(2\cancel{i})}$$

$$= \frac{1}{2}\left[\frac{1}{\cancel{i^2}+\cancel{1^2}+2i}\right] = \frac{1}{2} \times \frac{1}{2i} \times \frac{i}{i} = -\frac{i}{4}.$$

Residue at $(-i)$ $(m=1)$ $= \lim_{z \to -i} (z+i) f(z) = \lim_{z \to -i}\left[\cancel{(z+i)} \frac{z}{(z+1)^2\cancel{(z+i)}(z-i)}\right]$

$$= \lim_{z \to -i} \frac{z}{(z+1)^2(z-i)} = \frac{i}{4}.$$

(iv) $f(z) = \frac{z e^z}{(z-1)^3}$

POLE: Consider $\frac{1}{f(z)} = 0 \Rightarrow (z-1)^3 = 0 \Rightarrow z = 1, 1, 1.$ Triple pole

RESIDUE:

Residue at (1) $(m=3)$ $= \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2} (z-1)^3 f(z)$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2}\left[\cancel{(z-1)^3} \times \frac{z e^z}{\cancel{(z-1)^3}}\right]$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d}{dz}\left[z e^z + e^z (1)\right]$$

$$= \frac{1}{2} \lim_{z \to 1} \left[z \cdot e^z + e^z (1) + e^z\right]$$

$$= \frac{1}{2}\left[(1) e^1 + e^1 + e^1\right] = \frac{3e}{2}.$$

(v) $f(z) = \dfrac{\sin z}{(2z-\pi)^2}$

POLE: Consider $\dfrac{1}{f(z)} = 0 \Rightarrow (2z-\pi)^2 = 0$

$\Rightarrow z = \frac{\pi}{2}, \frac{\pi}{2}$ double pole.

RESIDUE: Residue at $\left(\frac{\pi}{2}\right)$ $(m=2)$ $= \dfrac{1}{1!} \lim\limits_{z\to\frac{\pi}{2}} \dfrac{d}{dz}\left[\left(z-\frac{\pi}{2}\right)^2 \cdot f(z)\right]$

$$= \lim_{z\to\frac{\pi}{2}} \frac{d}{dz}\left[\frac{\cancel{(z-\pi/2)^2} \cdot \sin z}{4\,\cancel{(z-\pi/2)^2}}\right]$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{d}{dz}\left[\frac{1}{4}\sin z\right]$$

$$= \lim_{z\to\frac{\pi}{2}} \left(\frac{1}{4}\cos z\right)$$

$$= \frac{1}{4}\cos\frac{\pi}{2} = \frac{1}{4}(0) = \underline{\underline{0}}$$

(vi) $f(z) = \dfrac{z+4}{(z-1)^2 (z-2)^3}$

POLE: $z = 1, 1$ double pole

$z = 2, 2, 2$, Triple pole

RESIDUE:

Residue at $(1) = -16$

Residue at $(2) = 16$

CAUCHY'S RESIDUE THEOREM.

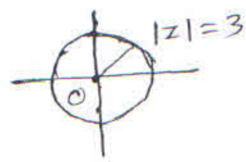
Let C be a simple closed curve and $f(z)$ be analytic within and on C except at a finite number of poles $a_1, a_2, \ldots a_n$ which lie inside C then

$$\int_C f(z)\,dz = 2\pi i\,(R_1 + R_2 + \cdots + R_n)$$

Where $R_1, R_2, \ldots R_n$ are the residues of $f(z)$ at $a_1, a_2, \ldots a_n$ respectively.

PROBLEMS

[1] Using the cauchy's residue theorem, Evaluate the integral $\int_C \frac{e^{2z}}{(z+1)(z-2)}\,dz$, where C is the circle $|z| = 3$.



sol: Given $f(z) = \frac{e^{2z}}{(z+1)(z-2)}$

POLE: Consider $\frac{1}{f(z)} = 0 \Rightarrow (z+1)(z-2) = 0$

$\Rightarrow z = -1$ and 2 are the simple poles and both lies inside $|z|=3$.

RESIDUE:

$$\text{Residue at } (-1) = \lim_{z\to -1} (z+1) f(z) = \lim_{z\to -1}\left[\cancel{(z+1)}\frac{e^{2z}}{\cancel{(z+1)}(z-2)}\right]$$

$$= \frac{e^{-2}}{-1-2} = \frac{e^{-2}}{-3} = R_1.$$

$$\text{Residue at } (2) = \lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2}\left[\cancel{(z-2)}\cdot\frac{e^{2z}}{(z+1)\cancel{(z-2)}}\right]$$

$$= \frac{e^4}{2+1} = \frac{e^4}{3} = R_2$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)(z-2)}\,dz = 2\pi i\,(R_1 + R_2) = 2\pi i\left[-\frac{e^{-2}}{3} + \frac{e^4}{3}\right]$$

—o—

[2] If C is the circle $|z| = 2$, Evaluate the integral $\int_C \frac{ze^z}{(z^2-1)} dz$ by using the residue theorem.

Sol: Here $f(z) = \frac{ze^z}{z^2-1} = \frac{ze^z}{(z-1)(z+1)}$

POLE: Consider $\frac{1}{f(z)} = 0 \Rightarrow (z-1)(z+1) = 0$

$\Rightarrow z = 1, -1$ are the simple poles and both lies inside $|z| = 2$.

RESIDUE:

Residue at $(1) = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \cancel{(z-1)} \cdot \frac{ze^z}{\cancel{(z-1)}(z+1)} = \frac{1 \cdot e^1}{2} = \frac{e}{2} = R_1$

Residue at $(-1) = \lim_{z\to -1} (z+1) f(z) = \lim_{z\to -1} \cancel{(z+1)} \frac{z \cdot e^z}{(z-1)\cancel{(z+1)}} = \frac{1 \cdot e^{-1}}{-1-1} = -\frac{e^{-1}}{2} = R_2$

By residue theorem,

$$\int_C \frac{ze^z}{z^2-1} dz = 2\pi i (R_1 + R_2) = 2\pi i \left(\frac{e}{2} - \frac{e^{-1}}{2}\right) = \pi i \left(e - \frac{1}{e}\right)$$

[3] By using the cauchy's Residue theorem, Evaluate the integral $\int_C \frac{z^2}{(z-1)^2(z+2)} dz$, where C is the circle $|z| = 5/2$.

Sol: Here $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

POLE: Consider $\frac{1}{f(z)} = 0$

$\Rightarrow (z-1)^2(z+2) = 0$

$\Rightarrow z = 1, 1 \rightarrow$ double pole

& $z = -2 \rightarrow$ simple pole

$\Big\}$ both lies inside $|z| = 5/2$

RESIDUE:

Residue at $(-2) = \lim_{z\to -2} (z+2) f(z) = \lim_{z\to -2} \cancel{(z+2)} \cdot \frac{z^2}{(z-1)^2 \cancel{(z+2)}}$

$$= \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9} = R_1$$

Residue at (1) $= \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \cdot f(z)\right]$
$(m=2)$

$$= \lim_{z\to 1} \frac{d}{dz}\left[\cancel{(z-1)^2} \cdot \frac{z^2}{\cancel{(z-1)^2}(z+2)}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left[\frac{z^2}{z+2}\right] = \lim_{z\to 1}\left[\frac{(z+2)(2z) - z^2(1)}{(z+2)^2}\right]$$

$$= \frac{(1+2)2(1) - 1^2}{(1+2)^2} = \frac{5}{9} = R_2$$

$$\therefore \int_C \frac{z^2}{(z-1)^2(z+2)} dz = 2\pi i\,(R_1 + R_2) = 2\pi i\left(\frac{4}{9}+\frac{5}{9}\right) = 2\pi i$$

[14] Using the cauchy's Residue theorem, Evaluate

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2 (z-2)} dz \quad \text{where } C : |z| = 3$$

Sol: Here $f(z) = \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2 (z-2)}$

POLE: Consider $\frac{1}{f(z)} = 0$

$\Rightarrow (z-1)^2 (z-2) = 0$

$\Rightarrow z = 1, 1 \rightarrow$ double pole $\}$ both Lies

$\& \; z = 2 \rightarrow$ simple pole $\}$ inside $|z|=3$

RESIDUE:

$\longrightarrow$

Residue at (1) $(m=2)$ $= \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 f(z)\right]$

$$= \lim_{z\to 1} \frac{d}{dz}\left[\cancel{(z-1)^2} \cdot \frac{\sin(\pi z^2)+\cos(\pi z^2)}{\cancel{(z-1)^2}(z-2)}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z-2)\{\cos(\pi z^2)2\pi z - \sin(\pi z^2)2\pi z\} - \{\sin \pi z^2 + \cos(\pi z^2)\}}{(z-2)^2}\right]$$

$$= \frac{(-1)\left[(\underbrace{\cos\pi}_{-1})2\pi - (\cancel{\sin\pi}^{\,0})2\pi\right] - \{\cancel{\sin\pi}^{\,0} + \underbrace{\cos\pi}_{=-1}\}}{(-1)^2}$$

$$= \frac{2\pi+1}{1^2} = 2\pi+1 = R_1$$

Residue at (2) $= \lim_{z\to 2} (z-2) f(z)$

$$= \lim_{z\to 2}\left[\cancel{(z-2)} \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^2 \cancel{(z-2)}}\right]$$

$$= \frac{\cancel{\sin(4\pi)}^{\,0} + \cancel{\cos(4\pi)}^{\,+1}}{(2-1)^2}$$

$\cos n\pi = \begin{cases} +1, & \text{even} \\ -1, & \text{odd} \end{cases}$

$\sin n\pi = 0$

$$= \frac{1}{1} = \boxed{1} = R_2$$

$$\therefore \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^2(z-2)}\,dz = 2\pi i(R_1+R_2) = 4\pi(\pi+1)i$$

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5 If C is the circle $|z| = 3/2$, Evaluate the following integrals (i) $\int_C \frac{e^{2z}}{(z+1)^3}\,dz$ (ii) $\int \frac{e^{2z}}{(z+1)^4}\,dz$ using the Residue Theorem.

Sol: (i) Here $f(z) = \frac{e^{2z}}{(z+1)^3}$

POLE: Consider $\frac{1}{f(z)} = 0 \Rightarrow (z+1)^3 = 0 \Rightarrow z = -1,-1,-1$, Triple pole

RESIDUE:

Residue at (-1) $(m=3)$ $= \frac{1}{2!} \lim_{z\to -1} \frac{d^2}{dz^2}\left[(z+1)^3 . f(z)\right]$

$$= \frac{1}{2} \lim_{z\to -1} \frac{d^2}{dz^2}\left[\cancel{(z+1)^3} . \frac{e^{2z}}{\cancel{(z+1)^3}}\right]$$

$$= \frac{1}{2} \lim_{z\to -1} \frac{d}{dz}\left[2e^{2z}\right]$$

$$= \frac{1}{2} \lim_{z\to -1} \left[4e^{2z}\right] = \frac{1}{2} 4 . e^{-2} = 2e^{-2} = R_1$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^3} dz = 2\pi i \; R_1 = 2\pi i . (2e^{-2}) = \frac{4\pi i}{e^2} .$$

~~Residue at~~

(ii) Here $f(z) = \frac{e^{2z}}{(z+1)^4}$

POLE: Consider $\frac{1}{f(z)} = 0 \Rightarrow (z+1)^4 = 0$
$\Rightarrow \boxed{z = -1, -1, -1, -1}$, pole of order 4.

RESIDUE:

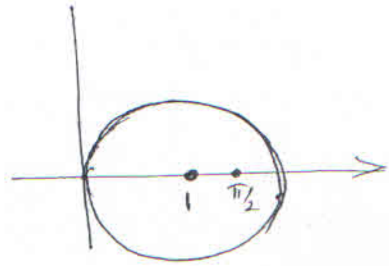
Residue at (-1) $(m=4)$ $= \frac{1}{3!} \lim_{z\to -1} \frac{d^3}{dz^3}\left[(z+1)^4 \; f(z)\right]$

$$= \frac{1}{6} \lim_{z\to -1} \frac{d^3}{dz^3}\left[\cancel{(z+1)^4} \frac{e^{2z}}{\cancel{(z+1)^4}}\right]$$

$$= \frac{1}{6} \lim_{z\to -1} \left[8e^{2z}\right] = \frac{1}{6} 8 \; e^{-2} = \frac{4}{3} e^{-2} = R_1$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4} dz = 2\pi i \; R_1 = \frac{8}{3} \frac{\pi i}{e^2}$$

$\boxed{6}$ Evaluate $\int_C \frac{z\cos z}{(z-\pi/2)^3}\,dz$ where $C: |z-1| = 1$ using Cauchy's Residue Theorem.



Sol: Here $f(z) = \frac{z\cos z}{(z-\pi/2)^3}$.

Pole: Consider $\frac{1}{f(z)} = 0 \Rightarrow (z-\frac{\pi}{2})^3 = 0$

$\Rightarrow z = \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \to$ Triple pole

RESIDUE: Residue at $\left(\frac{\pi}{2}\right)$ $(m=3)$

$$= \frac{1}{2!}\lim_{z\to\pi/2} \frac{d^2}{dz^2}\left[(z-\pi/2)^3 \cdot f(z)\right]$$

$$= \frac{1}{2}\lim_{z\to\frac{\pi}{2}} \frac{d^2}{dz^2}\left[\cancel{(z-\frac{\pi}{2})^3} \cdot \frac{z\cos z}{\cancel{(z-\frac{\pi}{2})^3}}\right]$$

$$= \frac{1}{2}\lim_{z\to\frac{\pi}{2}} \frac{d^2}{dz^2}\left[z\cos z\right]$$

$$= \frac{1}{2}\lim_{z\to\frac{\pi}{2}} \frac{d}{dz}\left[z(-\sin z) + \cos z\right]$$

$$= \frac{1}{2}\lim_{z\to\pi/2} \left[z(-\cos z) + (-\sin z)(1) - \sin z\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}\cancelto{0}{(-\cos\pi/2)} - \sin\frac{\pi}{2} - \sin\pi/2\right]$$

$$= \frac{1}{2}(-2) = \underline{\underline{-1}} = R_1$$

$$\therefore \int_C \frac{z\cos z}{(z-\pi/2)^3}\,dz = +2\pi i\, R_1 = -2\pi i$$

$\boxed{7}$ Evaluate $\int_C \frac{dz}{z^2(z^2+4)}$; $C: |z-2i| = 3$, using residue theorem.

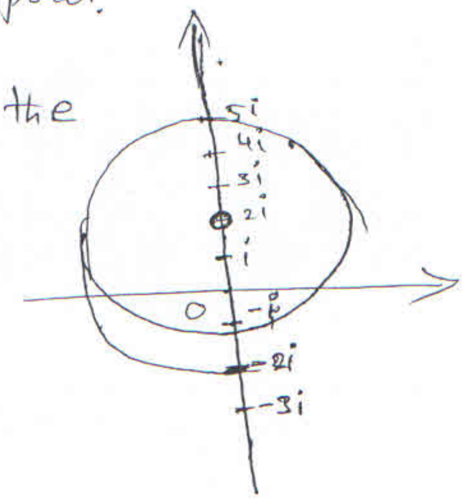
Sol: Here $f(z) = \frac{1}{z^2(z^2+4)} = \frac{1}{z^2(z+2i)(z-2i)}$

Pole: Consider $\frac{1}{f(z)} = 0 \Rightarrow z^2(z+2i)(z-2i) = 0$

$\rightarrow$

$\Rightarrow z = 0, 0 \longrightarrow$ double pole

& $z = -2i$ & $+2i \longrightarrow$ simple poles.



Of these poles $z=0$ & $2i$ Lies Inside the Circle $C : |z-2i| = 3$

$\therefore$ Residue at (0) $(m=2)$ $= \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz}\left[(z-0)^2 f(z)\right]$

$$= \lim_{z \to 0} \frac{d}{dz}\left[\cancel{z^2} \cdot \frac{1}{\cancel{z^2}(z^2+4)}\right]$$

$$= \lim_{z \to 0} \left[\frac{(z^2+4)(0) - 1(2z)}{(z^2+4)^2}\right] = \frac{0}{(4)^2} = 0 = R_1$$

Residue at $(2i)$ $(m=1)$ $= \lim_{z \to 2i} \left[(z-2i) \cdot f(z)\right]$

$$= \lim_{z \to 2i} \left[\cancel{(z-2i)} \cdot \frac{1}{z^2 (z+2i)\cancel{(z-2i)}}\right]$$

$$= \lim_{z \to 2i} \left[\frac{1}{z^2(z+2i)}\right]$$

$$= \frac{1}{4i^2(4i)} = \frac{1}{-16i} = \frac{1}{-16i} \times \frac{i}{i}$$

$$= \boxed{\frac{i}{16}} = R_2$$

$$\therefore \int_C \frac{dz}{z^2(z^2+4)} = 2\pi i (R_1 + R_2)$$

$$= \frac{\pi i^2}{8} = -\frac{\pi}{8}$$

$\boxed{8}$ By using Residue theorem Evaluate $\int_C \frac{2z^2+1}{(z+1)^2(z-2)} dz$ in the cases where (i) $C : |z|=3$

(ii) $C : |z+1| = 1$.

Sol: (i) Here $f(z) = \frac{2z^2+1}{(z+1)^2(z-2)}$ $C: |z| = 3$

POLE: Consider $\frac{1}{f(z)} = 0 \Rightarrow (z+1)^2(z-2) = 0$

$\Rightarrow z = -1, -1$ double pole, $\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ both lie inside C

& $z = 2$ simple pole

RESDUE:

Residue at (-1) $(m=2)$ $= \frac{1}{1!} \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 f(z)\right]$

$$= \lim_{z\to -1} \frac{d}{dz}\left[\cancel{(z+1)^2} \cdot \frac{2z^2+1}{\cancel{(z+1)^2}(z-2)}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(4z) - (2z^2+1)(1)}{(z-2)^2}\right]$$

$$= \frac{(-3)(-4) - (2(-1)^2+1)}{(-1-2)^2} = \frac{9}{9} = 1 = R_1$$

Residue at (2) $(m=1)$ $= \lim_{z\to 2} (z-2) f(z)$

$$= \lim_{z\to 2}\left[\cancel{(z-2)} \cdot \frac{2z^2+1}{(z+1)^2\cancel{(z-2)}}\right] = \lim_{z\to 2}\left[\frac{2z^2+1}{(z+1)^2}\right] = \frac{9}{9} = 1$$

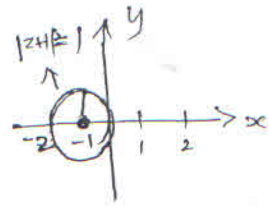
$= R_2$

$$\therefore \int_C \frac{2z^2+1}{(z+1)^2(z-2)} dz = 2\pi i\,(R_1 + R_2)$$

$$= 2\pi i\,(1+1)$$

$$= 4\pi i$$

(ii) Here $f(z) = \dfrac{2z^2+1}{(z+1)^2(z-2)}$; $|C| = |z+1| = 1$



POLE: Consider $\dfrac{1}{f(z)} = 0$

$\Rightarrow (z+1)^2(z-2) = 0$

$\Rightarrow z = -1, -1$ double pole $\rightarrow$ Lies inside C

& $z = 2$ Simple pole $\rightarrow$ Lies out side $|C| = |z+1| = 1$

RESIDUE:

Residue at (-1) $(m=2)$ $= \dfrac{1}{1!} \lim_{z \to -1} \dfrac{d}{dz}\left[(z+1)^2 f(z)\right]$

$= \lim_{z \to -1} \dfrac{d}{dz}\left[\cancel{(z+1)^2} \cdot \dfrac{2z^2+1}{\cancel{(z+1)^2}\, z-2}\right]$

$= \dfrac{9}{9} = 1 = R_1$ [Referen 8(i)]

$\therefore \displaystyle\int_C \frac{2z^2+1}{(z+1)^2(z-2)}\, dz = 2\pi i\,(R_1) = 2\pi i.$

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[9] Using the cauchy's Residue Theorem, Evaluate $\displaystyle\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\, dz$ in the following cases (i) $|z| = 4$ (ii) $|z-2| = 2$.

Sol: (i) Here $f(z) = \dfrac{3z^3+2}{(z-1)(z^2+9)}$, $C : |z| = 4$

$= \dfrac{3z^2+2}{(z-1)(z+3i)(z-3i)}$

POLE: Consider $\dfrac{1}{f(z)} = 0$

$\Rightarrow (z-1)(z^2+9) = 0$

$\Rightarrow z = 1, \pm 3i \rightarrow$ Simple poles & all lies inside $|z| = 4$.

RESIDUE: Residue at (1) $= \lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} \cancel{(z-1)} \dfrac{(3z^3+2)}{\cancel{(z-1)}(z^2+9)} = \dfrac{3(1)^3+2}{1^2+9}$

$= \dfrac{5}{10} = \dfrac{1}{2} = R_1.$

$$\text{Residue at } (+3i) = \lim_{z\to 3i} (z-3i)\cdot f(z)$$

$i = \sqrt{-1}$
$i^2 = -1$
$i^3 = i^2 \cdot i = -i$

$$= \lim_{z\to 3i} \cancel{(z-3i)}\cdot \frac{3z^3+2}{(z-1)(z+3i)\cancel{(z-3i)}}$$

$$= \lim_{z\to 3i} \frac{3z^3+2}{(z-1)(z+3i)} = \frac{3(3i)^3+2}{(3i-1)(6i)} = \frac{-3\times 27 i + 2}{18i^2 - 6i}$$

$$= \frac{-81i+2}{-18-6i} = -\frac{1}{6}\left[\frac{2-81i}{i+3} \times \frac{i-3}{i-3}\right]$$

$$= -\frac{1}{6}\left[\frac{2i - 162 i^2 - 6 + 243 i}{i^2 - 9}\right] = \frac{15+49i}{12} = R_2$$

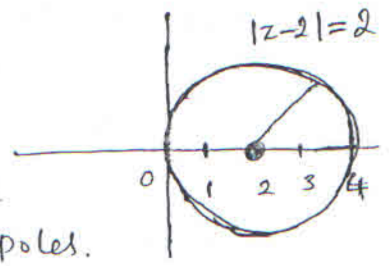
$$\text{Residue at } (-3i) = \lim_{z\to -3i} (z+3i)\cdot f(z)$$

$$= \lim_{z\to -3i} \cancel{(z+3i)}\cdot \frac{3z^3+2}{(z-1)\cancel{(z+3i)}(z-3i)}$$

$$= \lim_{z\to -3i} \frac{3z^3+2}{(z-1)(z-3i)} = \frac{15-49i}{12} = R_3$$

$$\therefore \int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz = 2\pi i\,(R_1+R_2+R_3) = 2\pi i\left(\frac{1}{2}+\frac{5}{2}\right) = 6\pi i.$$

(ii) Here $f(z) = \dfrac{3z^3+2}{(z-1)(z^2+9)}$ & $|C| = |z-2| = 2$



POLE: Consider $\dfrac{1}{f(z)} = 0 \Rightarrow z = 1$ simple pole
& $z = \pm 3i$ simple poles.

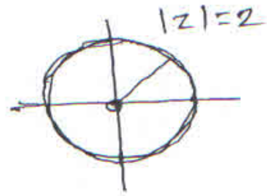
Of all these poles $z=1$ lies inside C & $z = \pm 3i$ lies outside C.

$$\underset{(m=1)}{\text{Residue at } 1} = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \cancel{(z-1)}\cdot\left(\frac{3z^3+2}{\cancel{(z-1)}(z^2+9)}\right) = \lim_{z\to 1} \frac{3z^3+2}{z^2+9}$$

$$= \frac{5}{10} = \frac{1}{2} = R_1$$

$$\therefore \int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz = 2\pi i\,(R_1) = \pi i$$

10 Evaluate $\int_C \tan z \, dz$ over $C : |z| = 2$.



Sol: Here $f(z) = \tan z = \dfrac{\sin z}{\cos z}$; $C : |z| = 2$

POLE: Consider $\dfrac{1}{f(z)} = 0 \Rightarrow \cos z = 0$

$$\Rightarrow z = \pm \pi/2, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$$

Of these poles $z = \pm \frac{\pi}{2}$ Lies inside C & all other Lies outside.

RESIDUE:

$$\text{Residue at } \left(\frac{\pi}{2}\right) = \lim_{z \to \pi/2} (z - \pi/2) \cdot f(z)$$

$$= \lim_{z \to \pi/2} (z - \pi/2) \cdot \frac{\sin z}{\cos z} = \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z \to \frac{\pi}{2}} \left[\frac{(z - \pi/2) \cos z + \sin z (1)}{- \sin z} \right] \text{ by L'Hospital Rule}$$

$$= \frac{0 + \sin \pi/2}{- \sin \pi/2} = -1 = R_1$$

$$\text{Residue at } \left(-\frac{\pi}{2}\right) = \lim_{z \to -\frac{\pi}{2}} (z + \pi/2) \cdot f(z)$$

$$= \lim_{z \to -\frac{\pi}{2}} \left(z + \frac{\pi}{2}\right) \cdot \frac{\sin z}{\cos z} = \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z \to -\frac{\pi}{2}} \left[\frac{(z + \pi/2) \cos z + \sin z (1)}{- \sin z} \right] \text{ by L'Hospitals Rule}$$

$$= \frac{0 + \sin(-\pi/2)}{- \sin(-\pi/2)} = \frac{-1}{1} = -1 = R_2$$

$$\therefore \int_C \tan z \, dz = 2\pi i (R_1 + R_2) = 2\pi i (-1 - 1) = -4\pi i$$

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